# Generalized entropies in dynamical systems

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*To my parents*

# Contents

# Preface

This thesis represents the results of my work in Groningen during the years 1995-2000. I first came to Groningen as a student of the Master Class on Dynamical Systems organized by the Mathematical Research Institute in the Netherlands. A program of the Master Class required participants to work on the so-called 'test problem'. Professor Floris Takens, who was supervising the Master Class students in Groningen, proposed me as the test problem to look at the family of the generalized entropies which he had previously suggested as the dynamical analogues of generalized dimensions. By that time the generalized dimensions had been already widely used not only in time series analysis, but were also successfully applied for the purposes of the multifractal analysis. The main idea of Floris Takens was to develop the corresponding theory of generalized entropies, and to see which of the 'dimension' results are valid in the entropy case, to investigate new features, and to develop methods for the numerical estimation of these entropies. The thesis closely follows this program, and gives answers to some of the questions raised.

After promising first results on the properties of the generalized entropies prof. Takens submitted a proposal to the Dutch Organization for Scientific Research (NWO) for a 4-year position for a PhD student. In September 1996 this position was granted, and in January 1997 I started to work in Groningen on this project.

One of my first tasks was to look at the so-called Rényi entropies which have been defined in terms very similar to the definition of the generalized entropies. The Rényi entropies have been mostly used in the physics literature, and it was claimed that these invariants provide new information about the dynamical system. However it turned out that these invariants, when treated from a rigorous mathematical point of view, do not actually contain any new information (when compared to the measure-theoretic entropy) for ergodic dynamical systems with positive entropy. Later requirements of ergodicity and positivity of entropy have been relaxed, yielding a similar result.

Approximately at the same time, when I started working on generalized entropies, Ya.B. Pesin and his co-authors in the series of papers proposed a general concept of the multifractality. Their formalism suggested a corresponding theory for local entropies. Generalized entropies, the main object of my interest, were obviously closely related to this problem, and that is why a great deal of this thesis is devoted to the multifractal analysis of local entropies. In particular, we have been able to establish the validity of the multifractal formalism for local entropies for a large class of dynamical systems.

From the very beginning of the project we have been aware of some examples of possible singularities in the family of the generalized entropies. One of these examples was a family of interval maps with an indifferent fixed point, i.e., maps expanding everywhere except one point, — the so-called Manneville-Pomeau maps. In order to understand the singularity in the family of the generalized entropies better and relate it to multifractal analysis, one had to understand the properties of the absolutely continuous invariant measures. The Manneville-Pomeau (MP) maps are very close in their properties to the uniformly expanding interval maps — a classical object in Dynamical Systems. Not surprisingly the properties of the corresponding absolutely continuous invariant measures of the MP maps are almost the same as those for the invariant measures of expanding maps. It is also known that in the expanding case, the absolutely continuous invariant measures are Gibbs. However this 'almost' results in the singularity in the family of the generalized entropies, while for the Gibbs measures the generalized entropies depend smoothly on the parameter.

Aernout van Enter introduced me to Christian Maes, who was interested in the 'almost' and 'weakly' Gibbs measures. During my subsequent visit to Leuven, together with Frank Redig and Annelies van Moffaert we were able to establish the weakly Gibbsian nature of the absolutely continuous invariant measure for the Manneville-Pomeau maps. It turns out the singularity in the family of the generalized entropies originated in the 'wrong large deviations' of the Manneville-Pomeau maps, and the same 'wrong large deviations' imply a non-Gibbsian nature of the invariant measures.

This thesis would not exist without help, advice, and support of a lot of people.

First of all I would like to thank my advisors Floris Takens and Henk Broer. Prof. Takens introduced me to this subject, and ever since he was a great source of encouragement, support, and inspiration for me. Except teaching and showing me how to do mathematics, Floris has also made a great effort in teaching me how to write about mathematics. I am very grateful to Floris for all the attention he has given to me and my research. I would like to thank Prof. Broer for his interest in my work, support, discussions, and valuable advice.

I would like to thank the members of the reading committee Prof. G. Keller, Prof. M.S. Keane and Prof. Ya.B. Pesin for their careful reading, remarks, and the approval of the manuscript.

During the whole period of my stay in Groningen, I had many useful discussions with professors H. Dehling, T. Mikosch, E. Thomas from the Department of Mathematics, and dr. A.C.D. van Enter from the Department of Physics.

In Moscow I would like to thank prof. B.M. Gurevich, prof. V.I. Oseledets, and dr. V.V. Ryzhikov for a constant interest in my work, advice and a possibility to present my results in a seminar 'Ergodic Theory and Statistical Physics' almost every time I visited Moscow during the last 4 years.

I have profited enormously from the discussions I had with dr. V. Baladi, dr. L. Barreira, prof. G. Ben Arous, prof. R. Burton, dr. T. Prellberg, prof. D. Ruelle, dr. J. Schmeling, prof. J.-P. Thouvenot, and prof. M. Urbanski. I am extremely grateful to them for answering my questions and sharing their insights.

The research, described in this thesis, has been supported by the Dutch Organization for Scientific Research (NWO), grant 613-06-551. I am extremely grateful to W.A.M Aspers and Dr. A.P.L. Zandee, both from NWO, for the management of this project. I would also like to acknowledge the financial support of my traveling expenses by PRODYN (Probabilistic Methods in Dynamical Systems) program of the European Science Foundation.

I am extremely grateful to prof. B. Braaksma and Annemieke Beereboom for their help at the beginning of my studies in Groningen.

On a personal level, I would like to thank my friends Gerton Lunter, Hans de Jong, Patricia Carvalho, Madeliene Niermeijer, and especially, Bojan and Jelena Basrak, who have been 'my family' here in Groningen for the last few years.

I will be always grateful to Rafail Kalmanovich Gordin, my school teacher of mathematics, without whose support and encouragement I would not dare to go and study mathematics in university.

This book is dedicated to my parents, dr. Elena Apeltsina and dr. Alexander Verbitski who were the first to show me that mathematics can be beautiful, although they are not professional mathematicians.

# Chapter 1

# Introduction

The theory of dynamical systems has undergone a dramatical revolution in the 20th century. The beauty and power of the theory of dynamical systems is that it links together different areas of mathematics and physics.

In the last 30 years a great deal of attention was dedicated to a statistical description of strange attractors. This led to the development of notions of various dimensions and entropies, which can be associated to the attractor, dynamical system or invariant measure.

In this chapter we review these notions and discuss relations between those, among which the most prominent is the so-called *multifractal formalism*.

At this moment, let us briefly discuss the general problem of multifractal analysis.

In the study of dynamical systems, and especially chaotic systems, sets with extremely complicated geometric structure occur quite naturally. Often these sets are regarded as *fractals*, meaning that these sets have some sort of self-similar structure. However, only few of these sets are truly self-similar. The following situation is more typical: there exists not one, but an infinite number of different so-called scalings which are intrinsic for the set. Such sets are called *multifractals*.

Various quantitative characteristics have been introduced for the global description of attractors and other invariant sets with complicated geometry. Examples of such quantities include various dimensions and entropies. At the same time, many of these global characteristics admit local or pointwise versions. This means that it is possible to define a dimension or an entropy at any given point. These quantities determine certain scalings at these points. What happens in the case of multifractals is the following. The range of possible values of pointwise dimensions (or entropies) is an uncountable bounded set. Often this set is an interval. One can decompose a set into subsets of points where the pointwise quantity (dimension or entropy) is constant. We can also say that each of these sets has a particular type of scaling. Multifractal analysis studies such decompositions, and describes

The chapter is based on F. Takens, E. Verbitski, *Multifractal analysis of dimensions and entropies*, preprint, 2000.

its geometry in terms of so-called multifractal spectra. One of the main aims of the multifractal formalism is to relate these multifractal spectra to the globally defined quantities like dimensions and entropies. The different scalings can also be interpreted as inhomogeneities in the geometry (in the case of dimensions) or the dynamics (in the case of entropies). In this respect, we can say that the multifractal analysis is concerned with such inhomogeneities, and provides their characterization.

Historically the multifractal formalism for dimensions has been studied first; quite recently, a more general setup was introduced, developing a concept of multifractality for a wide set of dynamic characteristics including entropy and Lyapunov exponents. In this thesis we will mainly concentrate on the analysis of entropies. In the case of dimensions, we recall existing results and compare them with our results [76, 73, 74, 75, 77, 79] for entropies. A striking difference, which we will emphasize, is that despite obvious similarities in formal definitions and concepts, rigorous results for the multifractal entropy spectra (see definition below) can be proven under substantially weaker hypothesis.

## 1.1 Basic concepts of Dynamical Systems

**Dynamics.** In this paper we will consider the following 3 types of dynamical systems:

- (measurable dynamics) $f: X \to X$ is a measure-preserving transformation of a Lebesgue space $(X, \mathcal{B}, \mu)$.
- (topological dynamics) $f: X \to X$ is a continuous transformation of a compact metric space $(X, d)$.
- (differentiable dynamics) $f: M \to M$ is a $C^k$-diffeomorphism, $k \geq 1$, of a smooth manifold $M$.

In the sequel we will often combine the above settings. For example, the first and the second, or the first and the third.

*Remark.* It is interesting to mention the relation between different types of the dynamical systems considered above. Firstly, any measure-preserving transformation can be represented in an equivalent (i.e., measure-theoretically isomorphic) way as a continuous transformation of a compact metric space preserving a Borel probability measure [19]. Secondly, any measure-preserving transformation can also be represented as a diffeomorphism of a two-dimensional torus preserving some Borel measure [38]. It is not known if in the latter case, the invariant measure can be made absolutely continuous with respect to the Lebesgue measure.

**Attractors.** Consider a transformation $f: X \to X$, where $X$ is a compact metric space or a smooth manifold, and where accordingly $f$ is a continuous transformation or a diffeomorphism. The *limit set* $\omega(x)$ of $x \in X$ is

$$\omega(x) = \{y \in X : \exists n_i \to \infty \text{ such that } \lim_{i\to\infty} f^{n_i}(x) \to y\}.$$

Suppose now that we are given some measure $m$ on $X$. We will call $m$ the *reference measure*. We do not assume $m$ to be invariant. In the case $f$ is a diffeomorphism of some open domain in $\mathbb{R}^n$, or $f$ is a diffeomorphism of a smooth Riemannian manifold, the Lebesgue measure is typically chosen as a reference measure. We say that a closed set $K \subseteq X$ is an attractor if the set

$$B(K) = \{x \in X : \omega(x) = K\},$$

called the basin of $K$, has positive $m$-measure, and there is no strictly smaller closed set $K' \subseteq K$ such that $B(K')$ coincides with $B(K)$ up to a set of measure 0. Attractors can be stationary, periodic, quasi-periodic (tori), or "strange". It is conjectured (see [49]) that a generic diffeomorphism has a finite number of attractors $K_1, \dots, K_N$ which attract almost every point, i.e., $m(X \setminus \cup_j B(K_j)) = 0$. However it is possible that there are no attractors at all (e.g., $f$ is the identity), or there may coexist infinitely many attractors [45].

We have to mention that many other definitions of attractors can be found in the literature, see e.g. [43] for a discussion. The above definition is an attempt to formalize a "naive" understanding of an attractor as a set of points to which a large set of points evolve under iterates of a dynamical system. Another useful property of the above definition is the so-called *indecomposability* property: an attractor can not contain any strictly smaller attractor.

*Example 1.1 (Solenoid).* This attractor appeared in the seminal paper [69] by S. Smale in 1967. Consider a solid torus $\mathbf{T}^2 = S^1 \times D = \{\varphi \in \mathbb{R} \mod 1\} \times \{z \in \mathbb{C} : |z| \le 1\}$ as a subset of $\mathbb{R}^3$. We define a diffeomorphism $f$ on $\mathbb{R}^3$, mapping the torus $\mathbf{T}^2$ into itself, by the following formula:

$$f(\varphi, z) = (2\varphi, \lambda z + \varepsilon e^{2\pi i \varphi}),$$

where $\lambda, \varepsilon > 0$ are such that $\lambda < \varepsilon$ and $\lambda + \varepsilon < 1$.

The torus $\mathbf{T}^2$ is mapped by $f$ into a solid tube which is wrapped twice inside $\mathbf{T}^2$. Note that the solid torus is expanded in the $\varphi$-direction, and is contracted in the $z$-directions. This, namely, continuous and complementary bundles of contracting and expanding directions, is typical for a *hyperbolic* dynamical system.

The solenoid is defined as the infinite intersection

$$K = \bigcap_{n \ge 0} f^n(\mathbf{T}^2).$$

Almost every point with respect to the Lebesgue measure on $\mathbf{T}^2$ belongs to the basin of attraction of $K$, and $K$ is an attractor in the above sense. However there exists a Lebesgue measure zero set of points $x$ such that $\omega(x) \subsetneq K$.

We would like to mention that a similar construction, as an example of a specific topological group, has also appeared in the PhD thesis of D. van Dantzig defended in Groningen in 1931.

A formal definition of hyperbolicity is due to Smale [69], and it clearly applies to the solenoid.

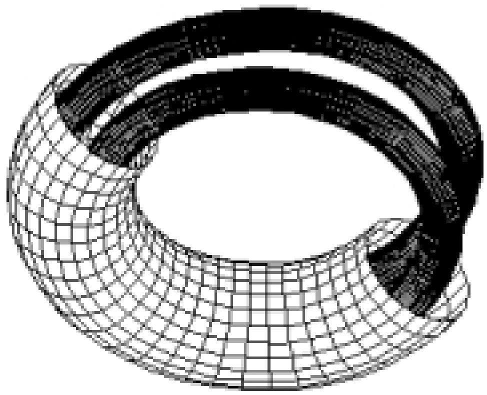

Figure 1.1: Solenoid in $\mathbb{R}^3$.

**Definition 1.2.** *For a given diffeomorphism $f$ of a smooth Riemannian manifold $M$, a compact set $\Lambda \subseteq M$, which is invariant, i.e. $f(\Lambda) = \Lambda$, is called* **hyperbolic** *if there exists a continuous decomposition*

$$T_x M = E_x^s \oplus E_x^u,$$

*such that*

$$(D_x f)E_x^s = E_{f(x)}^s, \ (D_x f)E_x^u = E_{f(x)}^u,$$

*and for some constants $C > 0$, $\lambda \in (0,1)$ (independent of $x$) the following holds*

$$||(D_x f^n)u|| \leq C\lambda^n ||u||, \ ||(D_x f^{-n})v|| \leq C\lambda^n ||v|| \tag{1.1}$$

*for every $x \in M$, $n \geq 0$ and all $u \in E_x^s$ and $v \in E_x^u$.*

Other well-known examples of attractors include the *Logistic* and *Hénon* attractors. However, these attractors are not hyperbolic.

**Statistical description of attractors.** Suppose $K \subseteq X$ is a compact attractor for $f : X \to X$ and some reference measure $m$. We say that a Borel probability measure $\mu$ on $K$ is a *Sinai–Ruelle–Bowen* (SRB) measure, (sometimes also called a *natural* or a *physical* measure), if there exists a set $B_0(K) \subseteq B(K)$ of full measure, i.e., $m(B(K) \setminus B_0(K)) = 0$, such that for every continuous function $\varphi : X \to \mathbb{R}$ and each $x \in B_0(K)$ one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int \varphi \, d\mu.$$

According to our definition, if an SRB measure exists, then it is unique. Nowadays the problem of existence of an SRB measure for various dynamical systems is an important mathematical question.

For hyperbolic attractors existence of the SRB measures was established by Ya. Sinai [67], D. Ruelle [58], R. Bowen and D. Ruelle [12]. Examples of attractors without SRB measures are known as well. A simple example of such sort was suggested by R. Bowen, and is now known as an *eye attractor*. The attractor $K$

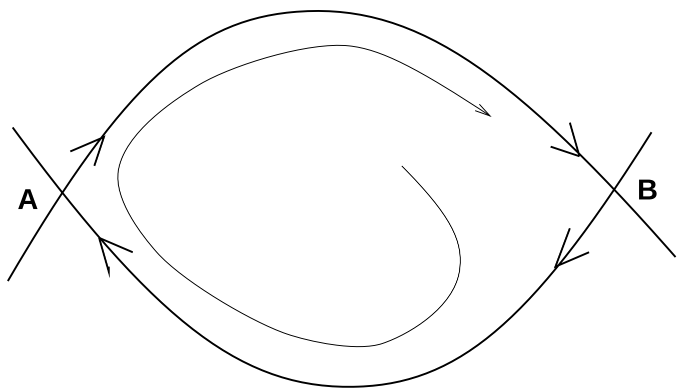


Figure 1.2: Bowen's example of an eye attractor.

consists of two saddle points $A, B$ and two separatrices, connecting them, see figure 1.2. The eigenvalues at the saddle points are assumed to be such that the loop $K$ is attracting. The basin of attraction $B(K)$ is a one-sided neighborhood of the loop. Any trajectory starting in the basin $B(K)$ oscillates between $A$ and $B$. This oscillation has the following property: for any function $\varphi$ such that $\varphi(A) \neq \varphi(B)$, one has

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \neq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)).$$

for every $x \in B(K)$. One can actually explicitly identify the lower and upper limits above in terms of eigenvalues at the two saddle points. This example was studied in great details by F. Takens in [72]. It also serves as a source for other abnormal examples, see [4]. Bowen's example has co-dimension 2 and is not persistent under small perturbations. It is also possible to construct examples of attractors without SRB measures in co-dimension 1, see [17]. It is an interesting question if such examples are possible in co-dimension 0, i.e., persistent in some sense.

For partially hyperbolic systems the question of existence of SRB measures has been studied by Ya.B. Pesin and Ya.G. Sinai in [54]. Recently, this investigation has been continued by J. Alves, C. Bonatti, and M. Viana in [2, 9] Also, the existence of SRB measures has been recently established by M. Benidicks and L.S. Young [7] for a large set of parameter values in the Hénon family.

**Gibssian approach.** In his classical paper Ya.G. Sinai [67] introduced the notion of Gibbs measures (states), inspired by the corresponding concepts in Statistical Physics, into Dynamical Systems. This approach proved to be extremely useful. It turns out that for hyperbolic dynamical systems the SRB measures are Gibbs measures as well. In the sequel we will see that Gibbs measures for certain dynamical systems allow for a complete multifractal analysis. A successful multifractal analysis of Gibbs measures is possible mainly due to the fact that one has a good control over the local structure of such measures. Another useful property of the Gibbs states is that they admit a global description as well; e.g. one can use the Variational Principles and the Large Deviations Principles, proved for a large class of dynamical systems and Gibbs states.

## 1.2 Dimensions and attractors

Very soon after the first examples of strange attractors emerged around 1970 it was realized that all these examples have a similar topological structure – a product of a smooth manifold with a Cantor set, see e.g. the collection of papers [68]. It also became apparent that new ways for describing the structure of attractors were needed. In this respect, the notion of Hausdorff dimension, introduced in the first quarter of 20-th century, proved to be of a great interest. The Hausdorff dimension is important for the description of sets, like these strange attractors. Within the further development of the dimension theory of dynamical systems some other dimension-like characteristics were introduced. In this section we recall some of them.

**Hausdorff dimension.** Consider a metric space $(X, d)$ and some subset $Z \subseteq X$. We say that an at most countable collection of open sets $\mathcal{U} = (U_i)$ is a $\delta$-cover of $Z$ if $\operatorname{diam}(U_i) < \delta$ for all $i$, and $Z \subseteq \cup_i U_i$. For $s \geq 0$ define an $s$-dimensional Hausdorff measure of $Z$ as follows

$$\mathcal{H}^s(Z) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(Z), \text{ where } \quad \mathcal{H}^s_\delta(Z) = \inf_{\mathcal{U}=(U_i)} \sum_i \big(\operatorname{diam}(U_i)\big)^s,$$

and the infimum is taken over all $\delta$-covers $\mathcal{U} = (U_i)$. It is not very difficult to show that there exists a value, $\dim_H(Z) = \hat{s}$, such that

$$\mathcal{H}^s(Z) = \begin{cases} +\infty, & \text{if } s < \dim_H(Z), \\ 0, & \text{if } s > \dim_H(Z). \end{cases} \tag{1.2}$$

The number $\dim_H(Z)$ is called the *Hausdorff dimension* of $Z$. The following properties of the Hausdorff dimension justify the use of the word dimension:

1) $\dim_H(\varnothing) = 0$, $\dim_H(\mathbb{R}^n) = n$;

2) if $Z_1 \subseteq Z_2$, then $\dim_H(Z_1) \le \dim_H(Z_2)$.

3) if $Z_1, \dots Z_k \subseteq X$, then

$$\dim_H\Big(\bigcup_{i=1}^{k} Z_i\Big) = \max_{i=1,\dots,k} \dim_H(Z_i).$$

Note however that not all these properties (e.g. 3) will be enjoyed by some other 'dimensions' defined below.

**Hausdorff dimension of a measure.** The Hausdorff dimension of an attractor $K$, being a static characteristic, does not take into account the dynamics. For a given dynamical system the SRB measure (if it exists) contains more information about the dynamics than the attractor as an invariant set. It turns out that one can associate various dimensions to measures as well. One of these notions – the Hausdorff dimension of a measure — proved to be extremely useful for the purposes of the statistical description of attractors.

Suppose we are given a probability measure $\mu$ with compact support. The Hausdorff dimension of the measure $\mu$ is defined as

$$\dim_H(\mu) = \inf\{\dim_H(A): \ \mu(A) = 1\}.$$

Clearly, $\dim_H(\mu) \le \dim_H(\mathrm{supp}\mu)$.

Now we give an example of a compact attractor which supports a lot of invariant measures, and these measures can be distinguished using the notion of the Hausdorff dimension of a measure. We will also use this example when we discuss the multifractal formalism.

*Example 1.3 (Skew tent map).* Consider the unit interval $I = [0,1]$, and for any $\lambda \in (0,1)$ define $f_\lambda : I \to I$ as follows

$$f_\lambda(x) = \begin{cases} x/\lambda, & x \in [0,\lambda] \\ (1-x)/(1-\lambda), & x \in (\lambda, 1]. \end{cases}$$

There is a natural partition $\xi_\lambda = \{I_0, I_1\} = \{[0,\lambda], (\lambda,1]\}$ associated to the map $f_\lambda$. For any $(j_0, \dots, j_n) \in \{0,1\}^n$ define a cylinder $I_{j_0,\dots,j_n}$ by

$$I_{j_0,\dots,j_n} = I_{j_0} \cap f_\lambda^{-1} I_{j_1} \cap \dots \cap f_\lambda^{-n} I_{j_n}.$$

Now for any $p \in (0,1)$ there exist a unique $f_\lambda$-invariant measure $\mu_p$ such that

$$\mu_p(I_{j_0,\dots,j_n}) = p^m (1-p)^{n+1-m},$$

where $m = \#\{k : j_k = 0\}$ — the number of zeros in the symbolic representation of the cylinder $I_{j_0,\dots,j_n}$. In fact, $\mu_p$ is the projection of the Bernoulli (or the product) measure with probabilities $(p, 1-p)$, defined on the set of all infinite sequences of 0's and 1's, onto the unit interval. The support of the measure $\mu_p$ is the whole interval $[0,1]$. The Hausdorff dimension of the measure $\mu_p$ is given by the following formula [22]:

$$\dim_H(\mu_p) = \frac{p\log p + (1-p)\log(1-p)}{p\log\lambda + (1-p)\log(1-\lambda)}.$$

The equality $\dim_H(\mu_p) = 1$ is possible if and only if $p = \lambda$. In this case $\mu_p$ is the Lebesgue measure on $I$.

The obvious relation between the Hausdorff dimension of a measure and its support motivates the following definition.

**Definition 1.4.** *Given a compact set $K$, we say that a measure $\mu$ with supp$(\mu) \subseteq K$ is a measure of maximal dimension if* $\dim_H(\mu) = \dim_H(K)$.

Of course, the problem of finding the measure of maximal dimension for a given compact $K$ does not make too much sense in general. Usually one would like to solve a more specific problem: given a transformation $f$ and an invariant set $K$, what is the maximal Hausdorff dimension of an invariant measure with a support in $K$? This is a difficult question about which not much is known. We refer to a survey paper [26] and later work [27, 39].

**Pointwise or local dimensions.** Consider a probability measure $\mu$ on a metric space $(X, d)$. Define the *lower and upper pointwise dimensions*

$$\underline{d}_\mu(x) = \liminf_{\varepsilon\to 0} \frac{\log\mu(\mathcal{B}(x,\varepsilon))}{\log\varepsilon}, \text{ and } \overline{d}_\mu(x) = \limsup_{\varepsilon\to 0} \frac{\log\mu(\mathcal{B}(x,\varepsilon))}{\log\varepsilon}.$$

If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ for some point $x \in X$, then we say that a local dimension at the point $x$ exists and denote it by

$$d_\mu(x) = \underline{d}_\mu(x) = \overline{d}_\mu(x).$$

A measure $\mu$ is called *exact dimensional* if there exists a constant $d$ such that $d_\mu(x) = \underline{d}_\mu(x) = \overline{d}_\mu(x) = d$ for $\mu-$a.e. $x \in X$. There are several known examples of invariant measures which are not exact dimensional, see [16, 40]. Recently it was shown [28] that a typical Borel probability measure on a compact metric space is not exact dimensional. In [36], a generic set of circle diffeomorphisms, admitting a unique invariant measure, was studied. It was shown that for a such diffeomorphism, the corresponding unique invariant measure is not not exact dimensional as well. L. Barreira, Ya.B. Pesin and J. Schmeling recently proved [6] the so-called Eckmann-Ruelle conjecture. This conjecture states that for any $C^{1+\alpha}$, $\alpha > 0$, diffeomorphism $f$ of a smooth Riemannian manifold, every hyperbolic measure is exact dimensional. We recall that the measure is called hyperbolic, if it is ergodic and the Lyapunov exponents of $f$ are non-zero $\mu$-almost everywhere.

Now let us come back to the problem of computing the Hausdorff dimension of a given set $Z$. This is usually a very difficult task. With the exception of a few simple examples (mainly Cantor sets) explicit calculations are not possible. For sets more general than Cantor sets the most powerful and widely used method is based on the so-called *Frostmann lemma.* This approach gives an estimate of the Hausdorff dimension of a set in terms of the pointwise dimensions of some measure concentrated on that set. We recall one result of such nature, proved by L.-S. Young [81].

**Theorem 1.5.** *Consider $Z \subseteq X$, and let $\mu$ be a finite Borel measure such that $\mu(Z) > 0$.*

*1) If $\underline{d}_\mu(x) \geq d$ for $\mu$-a.e. $x \in Z$ then $\dim_H(Z) \geq d$.*

*2) If $\overline{d}_\mu(x) \leq d$ for all $x \in Z$ then $\dim_H(Z) \leq d$.*

**Generalized dimensions.** One could also associate an infinite number of different dimensions to a given measure. There are two known approaches: one named after Rényi and another suggested by Hentschel-Procaccia.

**Rényi dimensions.** Suppose $\mu$ is a compactly supported measure in $\mathbb{R}^n$. For a given $\varepsilon > 0$ consider a partition of $\mathbb{R}^n$ into boxes of size $\varepsilon$:

$$\mathcal{P}_\varepsilon = \Big\{ P_{\mathbf{i}} = \prod_{k=1}^{n} [i_k\varepsilon, (i_k+1)\varepsilon) : \ \mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n \Big\}.$$

Following [34], see also [52], for $q > 1$ define the lower and upper Rényi dimensions of $\mu$ as

$$\underline{R}_q(\mu) = \liminf_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_{\mathbf{i}} \mu(P_{\mathbf{i}})^q}{\log \varepsilon},$$
$$\overline{R}_q(\mu) = \limsup_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_{\mathbf{i}} \mu(P_{\mathbf{i}})^q}{\log \varepsilon}.$$

The Rényi dimensions show the scaling (with respect to $\varepsilon$) of the $L_{q-1}(\mathbb{R}^n, \mu)$ norm of the following function

$$\xi(x) = \mu(P_{\mathbf{i}}(x)),$$

where $x \in \mathbb{R}^n$ and $P_{\mathbf{i}}(x)$ is the unique cell of the partition $\mathcal{P}$ containing $x$. The function $\xi(x)$ can be interpreted in the following way: $\xi(x)$ is the probability of the event that a $\mu$-random point $y$ lies in the same cell of our partition $\mathcal{P}_\varepsilon$ as $x$. Obviously,

$$\sum_{\mathbf{i}} \mu(P_{\mathbf{i}})^q = \int \left( \int \mathbb{I}(x, y \text{ are in the same cell }) \, d\mu(y) \right)^{q-1} d\mu(x),$$

where $\mathbb{I} : \mathbb{R}^n \times \mathbb{R}^n \to \{0, 1\}$ is the indicator function of the event.

For $q = 1$ we need a different definition. The following definition is suggested by a continuity argument:

$$\underline{R}_1(\mu) = \liminf_{\varepsilon \to 0} \frac{\sum_{\mathbf{i}} \mu(P_{\mathbf{i}}) \log \mu(P_{\mathbf{i}})}{\log \varepsilon},$$
$$\overline{R}_1(\mu) = \limsup_{\varepsilon \to 0} \frac{\sum_{\mathbf{i}} \mu(P_{\mathbf{i}}) \log \mu(P_{\mathbf{i}})}{\log \varepsilon}.$$

The Rényi dimensions of order 1 determine how the entropy of the partition $\mathcal{P}_\varepsilon$ scales with $\varepsilon$.

**Hentschel-Procaccia dimensions.** The one-parameter family of dimensions suggested in [34] basically uses the same motivation as the Rényi dimensions introduced above. However cubes of size $\varepsilon$ are substituted by balls of radius $\varepsilon$, which makes it possible to use these dimensions for spaces more general than $\mathbb{R}^n$. The Hentschel-Procaccia dimension of order $q$, $q > 1$, is defined as follows ([34, 52]):

$$\begin{aligned} \underline{HP}_q(\mu) &= \liminf_{\varepsilon\to 0} \frac{1}{1-q} \frac{\log \int \mu(\mathcal{B}(x,\varepsilon))^{q-1} d\mu}{\log \varepsilon}, \\ \overline{HP}_q(\mu) &= \limsup_{\varepsilon\to 0} \frac{1}{1-q} \frac{\log \int \mu(\mathcal{B}(x,\varepsilon))^{q-1} d\mu}{\log \varepsilon}, \end{aligned} \tag{1.3}$$

and for $q = 1$

$$\begin{aligned} \underline{HP}_1(\mu) &= \liminf_{\varepsilon\to 0} \frac{\int \log \mu(\mathcal{B}(x,\varepsilon)) d\mu}{\log \varepsilon}, \\ \overline{HP}_1(\mu) &= \limsup_{\varepsilon\to 0} \frac{\int \log \mu(\mathcal{B}(x,\varepsilon)) d\mu}{\log \varepsilon}. \end{aligned} \tag{1.4}$$

**Basic properties of generalized dimensions.** The generalized dimensions defined above are non-increasing functions of the parameter $q$. One can easily establish that

$$\underline{HP}_q(\mu) \le \underline{R}_q(\mu) \text{ and } \overline{HP}_q(\mu) \le \overline{R}_q(\mu)$$

for all $q \ge 1$. It was shown in [52] that for all $q > 1$

$$\underline{R}_q(\mu) = \underline{HP}_q(\mu), \quad \text{and} \quad \overline{R}_q(\mu) = \overline{HP}_q(\mu),$$

From now on we will understand $D_\mu(q)$ as one of the generalized dimensions of $\mu$ defined above. We prefer the choice of the lower Hentschel-Procaccia dimension, i.e., $D_q(\mu) = \underline{HP}_q(\mu)$. The motivation for this choice is as follows. The Hentschel-Procaccia dimensions can be defined for all metric spaces, while the Rényi dimensions are defined only for $\mathbb{R}^n$. Secondly we would like the generalized dimension $D_q(\mu)$ to be as small as possible, and the lower Hentschel-Procaccia dimension is the minimal among the four generalized dimensions defined above.

**Generalized dimensions of order $q < 1$.** Without loss of generality we can assume that $\mu$ is positive on open sets; otherwise, we restrict $\mu$ to its support. Formally, we can use the same formulae, for example (1.3), to define generalized dimensions for $q < 1$. It is, however, possible that for some $q_0$ the limits in (1.3) are equal to $+\infty$. Due to the monotonicity for all $q < q_0$ the situation will then be the same. In this case, we are left with a choice: either to say that the generalized dimensions of all orders $q$, $q \le q_0$, do not exist, or to accept the possibility of infinite dimensions. We tend to accept the latter. It is, however, a matter of taste. In all concrete cases, the generalized dimensions will be finite for all $q \in \mathbb{R}$.

However examples where $D_q$ becomes infinite, have been studied in the physics literature as well. For a simple example of this see [15]. There are a few other known examples of singular behavior of generalized dimensions $D_q$ with to respect to $q$, see [15, 46].

**Multifractal formalism.** The multifractal formalism, first suggested on the heuristic level in [78] and [32, 31], describes the relation between various dimension characteristics of measures. Let us describe this formalism in the form presently accepted in the mathematical literature [52]. Let $\mu$ be a Borel probability measure on some separable metric space $X$. Consider the following decomposition of $X$ in level sets of the function assigning to a point $x$ the pointwise dimensions of $\mu$ at $x$:

$$\begin{aligned} X &= \bigcup_{\alpha\in\mathbb{R}_+} K_\alpha \bigcup K_{no} \\ &:= \bigcup_{\alpha\in\mathbb{R}_+} \{x \in X :\ \underline{d}_\mu(x) = \overline{d}_\mu(x) = \alpha\} \bigcup \{x \in X :\ \underline{d}_\mu(x) \neq \overline{d}_\mu(x)\}. \end{aligned}$$

The multifractal dimension spectrum $\mathcal{D}_\mu(\cdot)$, by definition, is the function which assigns to each $\alpha$ the Hausdorff dimension of $K_\alpha$

$$\mathcal{D}_\mu(\alpha) = \dim_H(K_\alpha).$$

The *domain*, $\mathrm{dom}_D(\mu)$, of the dimension spectrum of $\mu$ is the set of all $\alpha$'s with a non-trivial set $K_\alpha$:

$$\mathrm{dom}_{\mathcal{D}}(\mu) = \{\alpha \geq 0 :\ K_\alpha \neq \varnothing\}.$$

**Definition 1.6.** *We say that the multifractal formalism is* valid *for the dimension spectrum of $\mu$ if*

*i) there exist $\underline{\alpha}$, $\overline{\alpha} \geq 0$ such that*

$$(\underline{\alpha}, \overline{\alpha}) \subseteq \mathrm{dom}_{\mathcal{D}}(\mu) \subseteq [\underline{\alpha}, \overline{\alpha}];$$

*ii) for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$ the following is true*

$$\mathcal{D}_\mu(\alpha) = \inf_{q\in\mathbb{R}} \big(q\alpha + T(q)\big),$$

*where $T(q) = (1-q)D_\mu(q)$;*

*iii) the dimension spectrum $\mathcal{D}_\mu(\alpha)$ is a smooth function of $\alpha$ on $(\underline{\alpha}, \overline{\alpha})$;*

Let us discuss the meaning of this definition. We require that the domain of the dimension spectrum is a finite interval. It can be open or closed from the right or left. Secondly we require some smoothness of the dimension spectrum. Known examples vary from a real-analytic to a continuous behavior. However for the moment we will understand "smooth" as being at least continuously differentiable. The second condition in the definition says that the dimension spectrum $\mathcal{D}_\mu(\alpha)$ is the *Legendre transform* of $T(q) = (1-q)D_\mu(q)$. Let us discuss the notion of the Legendre transform in greater details.

**Legendre transform.** A classical definition of the *Legendre-Fenchel transform* [57] is as follows. Suppose $f$ is a function defined on some interval $I$ which maybe finite or infinite. Its Legendre-Fenchel transform is the function $f^*$ given by

$$f^*(y) = \sup_{x\in I}\big(xy - f(x)\big) = -\inf_{x\in I}(f(x) - xy). \tag{1.5}$$

The Legendre-Fenchel transform $f^*(y)$ is a convex function on its domain $I^* = \{y : f^*(y) < +\infty\}$. Moreover, on the class of strictly convex functions the Legendre-Fenchel transform is invertible

$$f(x) = \sup_y\big(xy - f^*(y)\big),$$

and is also an involution, i.e., $f^{**} = f$. A pair of functions $(f, g)$ is said to be a Legenre-Fenchel pair if $g = f^*$ and $f = g^*$. Hence, $(f, f^*)$ is such a pair, if $f$ is strictly convex.

In the definition 1.6, $\mathcal{D}_\mu(\alpha)$ and $T(q)$ are required to satisfy the equality

$$\mathcal{D}_\mu(\alpha) = \inf_{q\in\mathbb{R}}\big(q\alpha + T(q)\big). \tag{1.6}$$

Formally, $\mathcal{D}_\mu(\alpha)$ is not a Legendre-Fenchel transform of $T(q)$, but the function $F(\alpha) = -\mathcal{D}_\mu(-\alpha)$ satisfies $F(\alpha) = T^*(\alpha)$, where $T^*(\alpha)$ is given by (1.5).

In order to avoid the unnecessary complications, it is an accepted practice in the multifractal literature to use a different definition of the Legendre transform:

$$f^*(y) = \inf_x\big(xy + f(y)\big). \tag{1.7}$$

According to such definition, (1.6) can be rewritten as

$$\mathcal{D}_\mu(\alpha) = T^*(\alpha).$$

Everywhere in the sequel, we use (1.7) as the definition of the Legendre transform. Observe also that according to (1.7), transform of a convex functions is a concave function.

**Validity of multifractal formalism.** It is rather obvious that such a strong statement can not be true in a great generality. Let us start with a simple example where the multifractal formalism is valid. In fact, this example motivated the above definition of multifractal formalism.

*Example 1.7 (Multifractal analysis of the skew tent maps).* Let $f$ be a skew tent map with $\lambda = 1/2$ and $\mu_p$ for $p \in (0,1)$ be as in Example 1.3. For simplicity we assume $p > 1/2$. It is not very difficult to see that the pointwise dimensions of $\mu_p$ can take any values between $-\log(p)/\log 2$ and $-\log(1-p)/\log 2$. Using a natural partition $\xi = \{I_0, I_1\} = \{[0,1/2], (1/2,1]\}$, one can show by a direct computation that the generalized dimensions are

$$D_\mu(q) = -\frac{1}{q-1}\frac{\log(p^q + (1-p)^q)}{\log 2}, \quad q \neq 1,$$
$$D_\mu(1) = -\frac{p\log p + (1-p)\log(1-p)}{\log 2}.$$

The family of generalized dimensions $D_\mu(q)$ depends real-analytically on $q$.

For every $x \in [0,1]$ there exists a unique infinite sequence $\mathbf{j} = (j_0, j_1, \dots)$ of zeros and ones such that

$$x = \bigcap_{n\geq 0} I_{j_0,\dots,j_n}.$$

Such sequence $\mathbf{j}$ is called the symbolic representation of $x$. Define (if the limit exists)

$$l(x) = \lim_{n\to\infty}\frac{1}{n}\#\{k:\ 1\leq k\leq n,\ j_k = 0\},$$

i.e., $l(x)$ is the asymptotic frequency of 0's in the symbolic representation of $x$. One readily checks that $d_{\mu_p}(x)$ exists if and only if $l(x)$ exists, and

$$d_{\mu_p}(x) = -\frac{l(x)\log p + (1-l(x))\log(1-p)}{\log 2}. \tag{1.8}$$

Using some large deviation results from probability theory, one can show [23] that indeed

$$\mathcal{D}_{\mu_p}(\alpha) = \inf_{q\in\mathbb{R}}\big(q\alpha + T(q)\big),$$

where $T(q) = (1-q)D_{\mu_p}(q)$. Hence, the multifractal formalism for $\mu_p$ is valid.

*Example 1.8 (Non-concave spectrum of local dimension).* If we consider mixtures of two measures $\mu_{p_1}$, $\mu_{p_2}$ with $p_1 \neq p_2$ from the above example, like

$$\mu = t\mu_{p_1} + (1-t)\mu_{p_2}, \quad t\in(0,1),$$

then for every $x$ such that $d_{\mu_1}(x)$, $d_{\mu_2}(x)$ exist, the pointwise dimension $d_\mu(x)$ exists as well and

$$d_\mu(x) = \min\{d_{\mu_1}(x), d_{\mu_2}(x)\}.$$

Hence, from (1.8) we conclude that $d_\mu(x)$ exists if and only if $l(x)$ exists. Finally, using the the spectra $\mathcal{D}_{\mu_1}(\alpha)$ and $\mathcal{D}_{\mu_2}(\alpha)$, we obtain the graph of $\mathcal{D}_\mu(\alpha)$, see figure 1.3. Therefore the dimension spectrum of $\mu$ is not a concave function, and hence it cannot be a Legendre transform of any function. Thus the multifractal formalism for such measures is not valid.

It is also possible to construct a measure for which the domain of the dimension spectrum is a union of two intervals, hence violating the first condition of our definition. For this it is sufficient to consider a measure which is a mixture of two measures with disjoint domains of their dimension spectra.

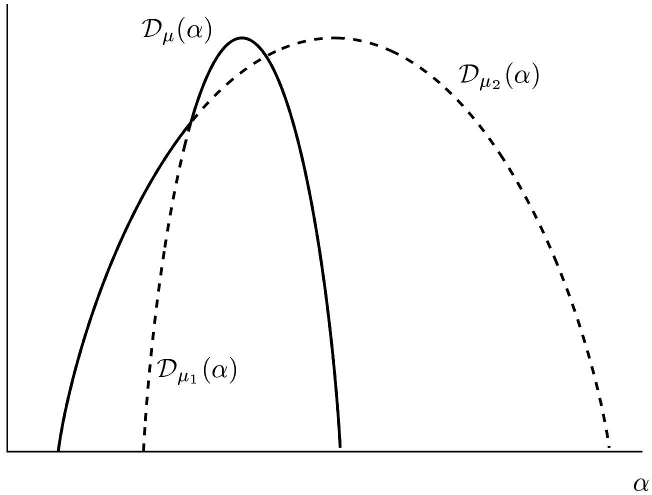


Figure 1.3: Example of a non-concave spectrum $\mathcal{D}_\mu(\alpha)$ (solid line) and the corresponding concave spectra $D_{\mu_i}(\alpha)$ (dashed lines).

An important class of measures is the class of the so-called *diametrically regular* or *doubling* measures. A measure $\mu$ is said to satisfy the doubling condition if there exists $A > 1$ such that for every sufficiently small $\varepsilon > 0$ and all $x$ one has

$$\mu(\mathcal{B}(x, 2\varepsilon)) \le A\mu(\mathcal{B}(x, \varepsilon)), \tag{1.9}$$

L. Olsen [46] showed that for *any* measure $\mu$, satisfying the doubling condition (1.9), the dimension spectrum $\mathcal{D}_\mu(\alpha)$ is bounded above by the Legendre transform of $T(q) = (1-q)D_\mu(q)$.

**Dimension spectrum for Gibbs measures, invariant under hyperbolic surface diffeomorphisms.** The above example 1.7 of a successful multifractal analysis of an invariant measure for one-dimensional expanding maps, raises a question: can the same be done in higher dimensions? The following result, due to D. Simpelaere [65, 66] and Ya.B. Pesin & H. Weiss [50, 51], shows that this is indeed the case, provided the dimension of the manifold is 2, and the dynamical system is hyperbolic.

**Theorem 1.9.** *Let $f : M \to M$ be a diffeomorphism of a compact smooth Riemannian surface $M$ and $\Lambda$ be a closed hyperbolic set for $f$. Let $\varphi : \Lambda \to \mathbb{R}$ be a Hölder continuous function and let $\mu$ be the corresponding Gibbs measure. Then the multifractal formalism is valid for the dimension spectrum of $\mu$. Moreover, the dimension spectrum $\mathcal{D}_\mu(\alpha)$ is real-analytic on $(\underline{\alpha}, \overline{\alpha})$.*

There are no results of such sort for diffeomorphisms on manifolds of higher dimensions.

One has to mention that successful multifractal analysis in higher dimensions is possible if one assumes that the dynamics is given by an *expanding conformal* map [50, 51] , i.e., a map $f : M \to M$ with

$$D_x f = a(x) I(x),$$

where $a(x) : M \to \mathbb{R}$ is a Hölder continuous function, such that $a(x) \geq \gamma > 1$ for some $\gamma$ and all $x$, and $I(x)$ is an isometry at every $x$. However, these assumptions, though multi-dimensional by nature, effectively reduce the problem to the one-dimensional setting considered above.

## 1.3 Entropy in Dynamical Systems

The notion of topological entropy for continuous transformations of compact metric spaces was introduced by R. Adler, A. G. Konheim and M. H. McAndrew in [1], later R. Bowen gave an equivalent definition using a different approach. One of the fundamental results, the so-called *Variational Principle*, establishes the equality between the topological entropy and the supremum of measure–theoretic (or Kolomogorov–Sinai) entropies over all invariant measures. The Variational Principle was first proved by E.I. Dinaburg [20] in a finite dimensional setting, and later generalized by W.L. Goodwyn [30] to any compact metric space; M. Misiurewicz [44] gave a short and very elegant proof.

It became apparent that the topological entropy is in fact a dimension-like characteristic of a dynamical system. R. Bowen [11] gave a definition of topological entropy of any set, not necessarily invariant or compact, along the lines of a standard definition of Hausdorff dimension, see (1.2). Later Ya.B. Pesin and B.S. Pitskel' [53] generalized this definition even further, allowing now the notion of topological pressure to be extended to a non-compact setting.

We recall another equivalent definition of topological entropy which can be found in [52]. Now we come to the formal definitions.

**Topological entropy.** Once again, let $(X, d)$ be a compact metric space, and let $f : X \to X$ be a continuous transformation. For any $n \in \mathbb{N}$ we define a new metric $d_n$ on $X$ as follows:

$$d_n(x, y) = \max\{d(f^k(x), f^k(y)) : \ k = 0, \dots, n-1\},$$

and for every $\varepsilon > 0$ we denote by $\mathcal{B}_n(x, \varepsilon)$ an open ball of radius $\varepsilon$ in the metric $d_n$ around $x$, i.e.,

$$\mathcal{B}_n(x, \varepsilon) = \{y \in X : \ d_n(x, y) < \varepsilon\}.$$

Suppose we are given some set $Z \subseteq X$. Fix $\varepsilon > 0$. We say that an at most countable collection of balls $\Gamma = \{\mathcal{B}_{n_i}(x_i, \varepsilon)\}_i$ covers $Z$ if

$$Z \subseteq \bigcup_i \mathcal{B}_{n_i}(x_i, \varepsilon).$$

For $\Gamma = \{\mathcal{B}_{n_i}(x_i, \varepsilon)\}_i$ put $n(\Gamma) = \min_i n_i$. Let $s \geq 0$ and define

$$M(Z, s, N, \varepsilon) = \inf_{\substack{\Gamma \text{ covers } Z \\ n(\Gamma) \geq N}} \sum_i \exp(-s n_i).$$

The quantity $M(Z, s, N, \varepsilon)$ does not decrease with $N$, hence the following limit exists

$$M(Z, s, \varepsilon) = \lim_{N \to \infty} M(Z, s, N, \varepsilon) = \sup_{N > 0} M(Z, s, N, \varepsilon).$$

It easy to show that there exists a critical value of the parameter $s$, which we will denote by $h_{top}(f, Z, \varepsilon)$, where $M(Z, s, \varepsilon)$ jumps from $+\infty$ to 0, i.e.,

$$M(Z, s, \varepsilon) = \begin{cases} +\infty, & s < h_{top}(f, Z, \varepsilon), \\ 0, & s > h_{top}(f, Z, \varepsilon). \end{cases}$$

As usual, there are no restrictions on the value $M(Z, s, \varepsilon)$ for $s = h_{top}(f, Z, \varepsilon)$. It can be infinite, zero, or positive and finite.

**Local (pointwise) entropy, the Brin-Katok formula.** Consider a compact metric space $(X, d)$. Let $f : X \to X$ be a continuous map and $\mu$ an invariant non-atomic Borel measure. Without loss of generality we may assume that $\mu$ is positive on open sets. In this case we define the lower (upper) local (pointwise) entropies as follows:

$$\underline{h}_\mu(f, x) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x, \varepsilon)), \tag{1.10}$$

$$\overline{h}_\mu(f, x) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x, \varepsilon)). \tag{1.11}$$

Note that the limits in $\varepsilon$ exist due to monotonicity.

We say that the local entropy exists at $x$ if

$$\underline{h}_\mu(f, x) = \overline{h}_\mu(f, x). \tag{1.12}$$

In this case the common value will be denoted by $h_\mu(f, x)$.

The following well-known result establishes the existence of the local entropies.

**Theorem 1.10 (Brin–Katok formula, [13]).** *Let $f : X \to X$ be a continuous map on a compact metric space $(X, d)$ preserving a non-atomic Borel measure $\mu$, then*

*1) for $\mu$-a.e. $x \in X$ the local entropy exists, i.e.,*

$$h_\mu(f, x) = \underline{h}_\mu(f, x) = \overline{h}_\mu(f, x);$$

*2) $h_\mu(f, x)$ is an $f$–invariant function of $x$, and*

$$\int h_\mu(f, x)\, d\mu = h_\mu(f),$$

*where $h_\mu(f)$ is the measure–theoretic entropy of $f$.*

*Remark.* An invariant measure $\mu$ is called ergodic if every invariant set has measure zero or one. If $\mu$ is ergodic, then every invariant function is constant almost surely. Hence if $\mu$ is ergodic, then $h_\mu(f,x) = h_\mu(f)$ for $\mu$-a.e. $x \in X$.

**Generalized entropies.** Similar to the case of generalized dimensions, see section 1.2, two families of generalized entropies were proposed. First we consider the Rényi entropies.

**Rényi entropies.** A formal definition of the Rényi entropy of order $q$ [21, 75] goes along the lines of the standard definition of the measure–theoretic entropy, except that Shannon's information function:

$$I(\mathbf{p}) = -\sum_i p_i \log p_i,$$

defined on a probability vectors $\mathbf{p} = (p_1, \ldots, p_n)$, $p_i \geq 0$, $\sum_i p_i = 1$, is replaced by a more general Rényi's information of order $q$, $q \neq 1$,

$$I_q(\mathbf{p}) = \frac{1}{1-q} \log \sum_i p_i^q.$$

Note that for $\mathbf{p}$ given, $I_q(\mathbf{p})$, with $I_1(\mathbf{p}) = I(\mathbf{p})$, depends continuously on q. The Rényi entropies of the measure-preserving dynamical systems are *measure-theoretic invariants*, i.e., two systems, isomorphic in the measure–theoretic sense, must have the same Rényi entropies of all orders. For ergodic dynamical systems these invariants do not contain any new information compared to the standard Kolmogorov–Sinai entropy, see [75, 80].

**Theorem 1.11 ([75, 80]).** *For an ergodic dynamical system $(X, \mu, f)$, the Rényi entropies are given by the following formula*

$$h_\mu(f,q) = \begin{cases} +\infty, & q < 1 \\ h_\mu(f), & q \geq 1. \end{cases} \tag{1.13}$$

However if $X = \mathbb{R}^n$, consideration of only the cubic partitions (like $\mathcal{P}_\varepsilon$ above) might provide additional information about the dynamics. Nevertheless, if one wishes the Rényi entropies to be measure-theoretic invariants, then one has to consider *all* possible partitions, arriving to (1.13), see chapter 3.

For non-ergodic dynamical systems the answer is slightly different ([74] and chapter 4 of this thesis), but nevertheless, the Rényi entropies are not very informative from a purely measure–theoretic point of view.

**Correlation entropies.** Another family of generalized entropies was proposed by F. Takens [71] as an entropy analogue of the Hentschel-Procaccia spectrum of generalized dimensions.

Let $(X, d)$ be a compact metric space, and $f : X \to X$ a continuous transformation preserving a Borel probability measure $\mu$. Without loss of generality we may assume that $\mu$ is positive on open sets. For each $q \in \mathbb{R}$, $q \neq 1$, we define the *lower and upper correlation entropies of order* $q$ as follows

$$\underline{H}_\mu(f,q) = \lim_{\varepsilon\to 0}\liminf_{n\to\infty} -\frac{1}{(q-1)n}\log\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu,$$
$$\overline{H}_\mu(f,q) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{(q-1)n}\log\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu,$$

and for $q = 1$:

$$\underline{H}_\mu(f,1) = \lim_{\varepsilon\to 0}\liminf_{n\to\infty} -\frac{1}{n}\int \log\mu(\mathcal{B}_n(x,\varepsilon))\,d\mu,$$
$$\overline{H}_\mu(f,1) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\int \log\mu(\mathcal{B}_n(x,\varepsilon))\,d\mu.$$

If for some $q \in \mathbb{R}$ one has $\underline{H}_\mu(f,q) = \overline{H}_\mu(f,q)$, we say that the correlation entropy $H_\mu(f,q)$ exists and let

$$H_\mu(f,q) = \underline{H}_\mu(f,q) = \overline{H}_\mu(f,q).$$

Let us recall some elementary properties of correlation entropies (see chapter 2):

1) for every $q_1 < q_2$ one has
$$\underline{H}_\mu(f,q_1) \geq \underline{H}_\mu(f,q_2) \geq 0, \quad \overline{H}_\mu(f,q_1) \geq \overline{H}_\mu(f,q_2) \geq 0;$$

2) $\underline{H}_\mu(f,1) = \overline{H}_\mu(f,1) = h_\mu(f)$, where $h_\mu(f)$ is the measure–theoretic entropy;

3) for $q > 1$ one has
$$\underline{H}_\mu(f,q) \leq \overline{H}_\mu(f,q) \leq h_\mu(f);$$

4) for $q \in [0,1]$ one has
$$h_\mu(f) \leq \underline{H}_\mu(f,q) \leq \overline{H}_\mu(f,q) \leq h_{top}(f),$$
where $h_{top}(f)$ is the topological entropy of $f$;

5) $\underline{H}_\mu(f,q)$ and $\overline{H}_\mu(f,q)$ depend continuously on $q$ for $q \in [0,1)$ and $q \in (1,\infty)$.

**Multifractal entropy spectrum.** Similar to the case of dimensions, consider the following decomposition of $X$:

$$X = \bigcup_{\alpha\in\mathbb{R}_+} K_\alpha \bigcup K_{no}$$
$$:= \bigcup_{\alpha\in\mathbb{R}_+} \{x \in X:\ \underline{h}_\mu(f,x) = \overline{h}_\mu(f,x) = \alpha\} \bigcup \{x \in X:\ \underline{h}_\mu(f,x) \neq \overline{h}_\mu(f,x)\}.$$

A multifractal entropy spectrum $\mathcal{E}_\mu(\cdot)$ assigns to each $\alpha$ the topological entropy of $K_\alpha$

$$\mathcal{E}_\mu(\alpha) = h_{top}(f, K_\alpha).$$

The domain of the entropy spectrum $\mathrm{dom}_{\mathcal{E}}(\mu)$, similar to the dimension case, is

$$\mathrm{dom}_{\mathcal{E}}(\mu) = \{\alpha : \quad K_\alpha \neq \varnothing\}.$$

**Multifractal formalism for the entropy spectrum.** The multifractal formalism for local entropies was first suggested on a heuristic level in [48, 29], a rigorous definition and the first results were obtained by L. Barreira, Ya.B. Pesin and J. Schmeling in [5].

As it was in the case of the dimension spectrum, we say that a multifractal formalism is valid for the entropy spectrum $\mathcal{E}_\mu(\alpha)$ if

*i)* there exist $\underline{\alpha}$, $\overline{\alpha} \geq 0$ such that

$$(\underline{\alpha}, \overline{\alpha}) \subseteq \mathrm{dom}_{\mathcal{E}}(\mu) \subseteq [\underline{\alpha}, \overline{\alpha}];$$

*ii)* for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$ the following is true

$$\mathcal{E}_\mu(\alpha) = \inf_{q \in \mathbb{R}} \big(q\alpha + T(q)\big),$$

where $T(q) = (1-q)H_\mu(q)$;

*iii)* the entropy spectrum $\mathcal{E}_\mu(\alpha)$ is a smooth function of $\alpha$ on $(\underline{\alpha}, \overline{\alpha})$.

**Entropy spectrum for Gibbs measures.** The first result on the validity of the multifractal formalism for the entropy spectrum in the case of Gibbs measures was obtained in [5]. The main interest of [5] was a multifractal analysis of the dimension spectrum, the result for the entropy spectrum was obtained as a corollary of the corresponding result for the dimension spectrum, and hence, is valid under the conditions of Theorem 1.9, i.e., in dimension 2.

In [77] we considered the entropy spectrum independent from the dimension spectrum and obtained the following result.

**Theorem 1.12.** *Let $f : M \to M$ be a diffeomorphism and $\Lambda$ be a closed hyperbolic set for $f$. Let $\varphi : \Lambda \to \mathbb{R}$ be a Hölder continuous function and $\mu$ be the corresponding Gibbs state. Then the multifractal formalism is valid for the entropy spectrum $\mathcal{E}_\mu$.*

In fact this result can be proven under much less restrictive conditions on the transformation $f$. It can be shown that the same result is true for expansive homeomorphism with the specification property (see definition 1.14 below) and Gibbs measures. In [77] it was shown that $\mathcal{E}_\mu(\alpha)$ is continuously differentiable in $\alpha$, $\alpha \in (\underline{\alpha}, \overline{\alpha})$; recall that the dimension spectrum of Gibbs measures invariant under 2-dimensional hyperbolic diffeomorphism is real-analytic.

As was done in the case of dimensions by considering mixtures of measures for which the multifractal formalism is valid, we can easily obtain new measures with non-concave entropy spectra, and hence the multifractal formalism can not be valid for such measures. Nevertheless it still makes sense to look at the Legendre transform of $T(q) = (1-q)H_\mu(q)$, since under some mild conditions on the measure this Legendre transform gives an upper estimate for the spectrum of local entropies.

**Theorem 1.13 ([76]).** *Let $f$ be a continuous transformation of a compact metric space $(X,d)$ with finite topological entropy. Consider an invariant probability non-atomic Borel measure $\mu$, which satisfies the following condition, called the weak entropy doubling condition: for every sufficiently small $\varepsilon > 0$ one has*

$$\lim_{n\to\infty} \frac{1}{n} \log \sup_x \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2))} = 0. \tag{1.14}$$

*Then there exist $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ such that*

*i)* $K_\alpha = \varnothing$ *for* $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$;

*ii) for* $\alpha \in (\underline{\alpha}, \overline{\alpha})$ *one has*

$$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) \le \inf_q (q\alpha - T(q)),$$

*where* $T(q) = (1-q)H_\mu(f,q)$.

## 1.4 Expansiveness and specification

In this section we discuss in greater details the properties of dynamical systems satisfying expansiveness and specification. These notions are not as widely used in Dynamical Systems as for example the notion of hyperbolicity. In what follows we stress similarities and differences between systems with these properties, and the more traditional hyperbolic dynamical systems.

**Specification.** The notion of specification was introduced to dynamical systems by R. Bowen in [10]. We recall now the definition of specification.

**Definition 1.14.** *We say that a homeomorphism $f : X \to X$ has the specification property (abbreviated to $f$ satisfies specification) if for each $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that the following holds:*
*if*

*a)* $I_1, \dots, I_k$ *are intervals of integers,* $I_j \subseteq [a,b]$ *for some* $a, b \in \mathbb{Z}$ *and all* $j$,

*b)* $\operatorname{dist}(I_i, I_j) \ge m(\varepsilon)$ *for* $i \ne j$,

*then for arbitrary* $x_1, \dots, x_k \in X$ *there exists a point* $x \in X$ *such that*

*1)* $f^{b-a+m}(x) = x$,

*2)* $d(f^n(x), f^n(x_j)) < \varepsilon$ *for* $n \in I_j$.

The meaning of this definition is as follows: given $\varepsilon > 0$ and any finite number of pieces of orbits,sufficiently separated in time, one can find a periodic point, which $\varepsilon$-shadows the specified pieces of orbits.

Various modifications of the above definition appeared in the literature as well. Among the most important are the following:

1) [64, 19] Specification is restricted to 2 intervals, i.e., $k = 2$.

2) [60, 33] Assumption that the shadowing point $x$ is periodic is dropped.

3) [42] The switching time $m = m_\varepsilon$ is no longer assumed uniform: it can depend on the lengths of intervals $I_1, \dots, I_k$, but not on the points $x_1, \dots, x_k$. Moreover $m_\varepsilon(\cdot)$ must be small compare to the $\max(|I_1|, \dots, |I_k|)$, in the sense that
$$\lim_{\max(|I_1|,\dots,|I_k|)\to\infty} \frac{m_\varepsilon(|I_1|, \dots, |I_k|)}{\max(|I_1|, \dots, |I_k|)} = 0.$$

It is well-known that Axiom A (and therefore Anosov) systems satisfy specification, e.g. see [37]. In [42] B. Marcus introduced the above variant 3 and showed that ergodic (not necessarily hyperbolic!) toral automorphisms satisfy this version of weak specification.

The specification property is rather strong. Transformations with specification are *topologically mixing*, i.e., for any two open sets $U, V$ one has $T^{-n}U \cap V \neq \varnothing$ for all sufficiently large $n$. Moreover, for continuous transformations of compact metric spaces with specification, measures concentrated on periodic points are dense in the set of all invariant measures, see e.g. [19].

Surprisingly for continuous transformations on the interval the following holds.

**Theorem 1.15 ([8]).** *A continuous transformation of the interval satisfies specification if and only if it is topologically mixing.*

Topological mixing is not exceptional even for non-hyperbolic systems. For example, from the results by M. Jakobson [35] we know that in the logistic family $f_r(x) = rx(1-x)$ topological mixing holds for a set of parameters of positive Lebesgue measure.

However the next result shows that Theorem 1.15 is not true for piecewise continuous transformations of the interval.

**Theorem 1.16 ([14, 62]).** *Consider a family of piecewise monotonic transformations $\{f_\beta\}$, $\beta > 1$, called $\beta$-shifts, and given by*
$$f_\beta(x) = \beta x \mod 1.$$
*Then the set of parameters $\beta$ for which $f_\beta$ satisfies specification is dense, has Lebesgue measure $0$ and Hausdorff dimension $1$.*

The so-called *Markov property* of piecewise continuous transformations of the interval is often used in literature, and is, for example, crucial for the existence of ergodic absolutely continuous invariant measures in the case of expanding maps. The Markov property consists of the following. Consider a piecewise monotonic tansformation $f$ of the interval $I$ and let $\{I_1, \dots, I_M\}$ be the corresponding partition into the intervals of monotonicity. Then $f$ is said to have a Markov property if for every $j = 1, \dots, M$, the image $f(I_j)$ is a union of some $I_k$'s.

The above result on the specification property of $\beta$-shifts should be compared to another well-known fact (see [14, 62]): for the family $\{f_\beta\}$ as above, the set of parameter values for which $f_\beta$ has a *Markov property* is at most countable, and hence has measure and the Hausdorff dimension zero. Thus we can say that for a family of $\beta$-shifts, the specification property is more typical than the Markov property.

It is also interesting to mention that J. Schmeling formulated the following conjecture.

**Conjecture.** In the family of $\beta$-shifts $\{f_\beta\}$, $\beta > 1$, weak specification in the sense of Marcus (variant 3 above) is a generic property in the measure-theoretic sense, i.e., it holds for Lebesgue almost all $\beta > 1$.

**Expansiveness.** Another important property of a dynamical system which is closely related to the sensitive dependence on initial conditions is the so-called expansiveness.

**Definition 1.17.** *An invertible transformation $f : X \to X$ of a metric space $(X, d)$ is called* expansive *if there exists a constant $\gamma > 0$ such that for any $x, y \in X$, $x \neq y$, there exists $n \in \mathbb{Z}$ with*

$$d(f^n(x), f^n(y)) \geq \gamma. \tag{1.15}$$

*If $f : X \to X$ is not invertible, we say that $f$ is* positively expansive *if there exists $\gamma > 0$ such that for any $x, y \in X$, $x \neq y$, one can find $n \in \mathbb{Z}_+$ satisfying (1.15).*

This definition corresponds to the intuitive understanding of chaotic dynamics: different points, eventually, will be separated by the dynamics. A similar, but stronger, property can be formulated:

**Definition 1.18.** *A map $f : X \to X$ on a metric space $(X, d)$ is called* (locally) expanding *if there exists $\lambda > 1$ and $\varepsilon_0 > 0$ such that for all $x, y \in X$, with $d(x, y) < \varepsilon_0$, one has*

$$d(f(x), f(y)) \geq \lambda d(x, y).$$

It is clear that expanding transformations are positively expansive. One could also easily construct examples of positively expansive, but not expanding transformations. Nevertheless, these notions are somewhat equivalent, as the next statement shows.

**Theorem 1.19 ([56]).** *Suppose $(X, d)$ is a compact metric space, and $f : X \to X$ is a continuous transformation which is positively expansive. Then there exists a metric $d'$ on $X$ compatible with $d$ (i.e., the topology generated on $X$ by $d$ and $d'$ are the same) such that $f$ is expanding on $(X, d')$.*

*Remark.* The above result states that there are actually no truly positively expansive, but not expanding transformations: one can always get expanding dynamics by considering a different metric. However the new metric $d'$ might not be *equivalent* to the original metric $d$. This might cause some problems. For example, it is well known, that the topological pressure $P(f, \varphi)$ of a continuous transformation $f : X \to X$ and a continuous function $\varphi : X \to \mathbb{R}$, though defined using the metric, is independent of this metric. Nevertheless the metric plays an important role in the question of existence and uniqueness of Gibbs states. One usually restricts oneself to a certain class of potentials which is strictly smaller than the space of all continuous functions. A typical example is the class of all Hölder continuous functions. It is not clear that such a class of functions may not be preserved when the metric is changed in a compatible, but nonequivalent way.

Markov partitions are widely used in dynamical systems, see e.g. [37]. However results, establishing existence of finite Markov partitions, are mainly known for hyperbolic dynamical systems. In this respect the following result might be of some interest.

**Theorem 1.20 ([18]).** *Let $f : X \to X$ is a continuous transformation of a compact metric space. Suppose that $f$ is positively expansive and open, i.e. open sets are mapped to open sets by $f$. Then there exists a finite Markov partition of arbitrarily small diameter.*

The overall picture for expansive homeomorphisms is slightly different. This is not very surprising: two directions in time make possible a more intricate geometric behavior. Nevertheless, results similar to Theorems 1.19, 1.20 were established for expansive systems as well. Before we proceed with further exposition, we would like to recall some definitions.

**Definition 1.21.** *For a homeomorphism $f$ of a compact metric space $(X, d)$ we define* local stable *and* unstable sets *of $x \in X$ as follows: for $\varepsilon > 0$ put*

$$W_f^{s,\varepsilon}(x,d) = \bigl\{y \in X : \ d(f^n(x), f^n(y)) \leq \varepsilon \ \text{ for all } n \geq 0\bigr\},$$
$$W_f^{u,\varepsilon}(x,d) = \bigl\{y \in X : \ d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon \ \text{ for all } n \geq 0\bigr\}.$$

The following result of K. Sakai [61] shows that by changing metric we can achieve "usual" hyperbolic behavior on stable and unstable sets.

**Theorem 1.22 ([61]).** *Let $f$ be an expansive homeomorphism of a compact metric space $(X, d)$. Then there exists a compatible metric $\bar{d}$ on $X$, $\lambda \in (0, 1)$ and $\varepsilon > 0$ such that for each $x \in X$ and any $n \geq 0$ one has:*

$$\bar{d}(f^n(x), f^n(y)) \leq \lambda^n \bar{d}(x,y) \quad \text{ for all } \ y \in W_f^{s,\varepsilon}(x, \bar{d}), \tag{1.16}$$
$$\bar{d}(f^{-n}(x), f^{-n}(y)) \leq \lambda^n \bar{d}(x,y) \quad \text{ for all } \ y \in W_f^{u,\varepsilon}(x, \bar{d}). \tag{1.17}$$

Estimates (1.16), (1.17) are analogous to estimates (1.1) for hyperbolic diffeomorphisms. Nevertheless these estimates by themselves are not enough to guarantee the existence of Markov partitions. For this we have to require more: namely, the so-called *local product structure* defined below. Homeomorphisms satisfying (1.16), (1.17) and having a local product structure appeared in the literature under different names. For example, in [59] D. Ruelle calls such homeomorphisms together with the underlying space $X$ a *Smale space*, in [41] R. Mañé calls them *hyperbolic homeomorphisms*, and in [3] such homeomorphisms are called *topological Anosov homeomorphisms* (TAH). All these notions were proved to be equivalent, see [47].

**Definition 1.23 ([41]).** *A homeomorphism $f$ of $(X,d)$ is called* hyperbolic, *if there exist $\varepsilon > 0$, $\lambda \in (0,1)$ such that*

*1) on $W_f^{s,\varepsilon}(x,d)$, $W_f^{u,\varepsilon}(x,d)$ the estimates* (1.16), (1.17) *are true;*

*2) (local product structure) for any $x, y \in X$ with $d(x,y) < \varepsilon$, one has*

$$\#\left(W_f^{s,\varepsilon}(x,d) \cap W_f^{u,\varepsilon}(y,d)\right) = 1.$$

Hyperbolic homeomorphisms admit finite Markov partitions. Moreover, in complete analogy with Anosov systems, hyperbolic homeomorphisms are structurally stable. One has to say that there are examples of diffeomorphisms which are not Anosov, but are topological Anosov homeomorphisms, i.e., hyperbolic homeomorphisms [3].

Also, for a hyperbolic homeomorphism the topological pressure $P(f,\cdot) : C^\alpha(X) \to \mathbb{R}$, where $C^\alpha(X)$ is the set of Hölder continuous functions with Hölder constant $\alpha$, is real-analytic [59]. The latter becomes important for possible applications to the multifractal analysis.

We have seen (Theorem 1.20) that for positively expansive maps the fact that the map is open was sufficient for the existence of a finite Markov partition. For expansive homeomorphisms one has to require more. J. Ombach in [47] showed that for expansive homeomorphisms the *pseudo-orbit tracing property* (POTP) is equivalent to the local product structure, i.e., the second condition in the definition 1.23. We recall the definition of the POTP.

**Definition 1.24.** *A sequence of points $\{x_n\}_{n\in\mathbb{Z}}$ is called a $\delta$-pseudo orbit if*

$$d(f(x_n), x_{n+1}) \le \delta \quad \textit{for all } n \in \mathbb{Z}.$$

*We say that a homeomorphism $f$ has a pseudo-orbit tracing property (also called shadowing) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that any $\delta$-orbit can be $\varepsilon$-shadowed by some orbit, that is, there exists $x \in X$ satisfying*

$$d(x_n, f^n(x)) \le \varepsilon \quad \textit{for all } n \in \mathbb{Z}.$$

Combining the results by K. Sakai (Theorem 1.22) and J. Ombach [47] we obtain the following statement.

**Theorem 1.25.** *Let $f$ be a homeomorphism of a compact metric space $(X,d)$. Then the following conditions are equivalent:*

*1) $f$ is expansive and has the pseudo-orbit tracing property;*

*2) $f$ is a hyperbolic homeomorphism on $(X,\bar{d})$ for some compatible metric $\bar{d}$.*

Expansiveness and positive expansiveness are rather strong properties of dynamical systems. There exist no expansive homeomorphisms on 1-dimensional manifolds. If a 2-dimensional manifold $M$ admits an expansive homeomorphism $f$, then $M$ must be a two-torus $\mathbb{T}^2$ and $f$ is conjugated to an Anosov diffeomorphism. Also, on some 3-dimensional manifolds a similar classification is possible.

## 1.5 Concluding remarks

In this chapter we have considered two multifractal spectra: for local dimensions and entropies, and compared results on the validity of the multifractal formalism. We observed that the results for entropies are valid under milder assumptions on the underlying dynamics, when compared to the corresponding results for dimensions.

The main problem in establishing the results on the validity of the multifractal formalism for local dimensions for higher dimensional dynamical systems, lies in the following unpleasant, and not yet understood, phenomenon [24, 25, 55, 70]: There exist examples of families of hyperbolic dynamical systems in dimensions 3 and higher such that the dimension of the attractors depends in a discontinuous way on the parameters. Moreover, the set of parameters can be decomposed into two sets. The first one is a large set of parameters, for which the dimension of the corresponding attractor is given by the same formula (the so-called Bowen–Ruelle equation) as it was for two dimensional systems. We may say that for these parameters, the dimension of the attractor is equal to what we have expected. The second set is complementary to the first set, is much smaller, and for parameters from this set, the dimension of the attractor is strictly smaller than the number, predicted by the Bowen-Ruelle equation. We have to stress that the Bowen-Ruelle equation is fundamental in the present treatment of the dimensions of attractors and the multifractal analysis of invariant measures.

One can ask, however, about the validity of the multifractal formalism not for all systems, but for those corresponding to a set of "good" parameters. Some results in this direction were obtained in [63]. However at the present moment we are quite far from answering this question.

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# Chapter 2

# Correlation entropies

In this chapter we introduce the notion of correlation entropies. We establish their basic properties such as monotonicity, continuity, etc. Also we obtain the estimates for correlation entropies in terms of other well-known invariants such as topological and measure-theoretic entropies. We derive explicit expressions for the correlation entropies for shift invariant Bernoulli and Gibbs measures on symbolic spaces. Finally, we also discuss possible singularities in the spectrum of the correlation entropies and their relation to non-uniformly hyperbolic behavior of dynamical systems.

This chapter is of an introductory nature, and most of the specific examples from this chapter will be extended later to more general settings.

## 2.1 Definition and basic properties

Let $(X, d)$ be a compact metric space and $f : X \to X$ be a continuous transformation preserving a Borel probability measure $\mu$. We will always assume that $\mu$ is positive on open sets. For every $n \in \mathbb{N}$ and any $\varepsilon > 0$ we define

$$\begin{aligned} \mathcal{B}_n(x, \varepsilon) &= \left\{ y \in X : \ d(f^i(x), f^i(y)) < \varepsilon \ \text{ for all } \ i = 0, \dots, n-1 \right\} \\ &= \left\{ y \in X : \ d_n(x, y) < \varepsilon \right\}, \end{aligned}$$

where $d_n(x, y) = \max\limits_{i=0,\dots,n-1} d(f^i(x), f^i(y))$.

For any $q \in \mathbb{R}$, $q \neq 1$, every $\varepsilon > 0$ and $n \in \mathbb{N}$ we define the correlation integral

$$I(q, \varepsilon, n) = -\frac{1}{(q-1)n} \log \int \mu\big(\mathcal{B}_n(x, \varepsilon)\big)^{q-1} \, d\mu. \tag{2.1}$$

For $q = 1$, we define $I(1, \varepsilon, n)$ by

$$I(1, \varepsilon, n) = -\frac{1}{n} \int \log \mu\big(\mathcal{B}_n(x, \varepsilon)\big) \, d\mu. \tag{2.2}$$

Our main task in this chapter will be the study of the asymptotic behavior of $I(q, \varepsilon, n)$ as $n \to \infty$ and $\varepsilon \to 0$. We start with the following simple observation.

**Lemma 2.1.** *The family of correlation integrals, defined by (2.1), (2.2), has the following monotonicity properties*

1) $I(q,\varepsilon_1,n) \geq I(q,\varepsilon_2,n)$ *for any* $0<\varepsilon_1<\varepsilon_2$;

2) $I(q_1,\varepsilon,n) \geq I(q_2,\varepsilon,n)$ *for any* $q_1<q_2$, *such that* $q_1,q_2 \neq 1$.

*Proof.* (1) Obvious.

(2) We recall the Lyapunov inequality [7]: for any function $h: X \to \mathbb{R}$ and any $0<s<t$

$$\left(\int |h|^s d\mu\right)^{\frac{1}{s}} \leq \left(\int |h|^t d\mu\right)^{\frac{1}{t}}.$$

Using the Lyapunov inequality we immediately obtain the second statement for $q_1<q_2<1$ and $1<q_1<q_2$. Suppose now that $q_1<1<q_2$. Put $\delta := \min\{1-q_1, q_2-1\}$. Then for $q'=1-\delta$ and $q''=1+\delta$ one has

$$\begin{aligned} I(q',\varepsilon,n)-I(q'',\varepsilon,n) &= \frac{1}{\delta n}\log\left[\int \mu\left(\mathcal{B}_n(x,\varepsilon)\right)^{-\delta} d\mu \times \int \mu\left(\mathcal{B}_n(x,\varepsilon)\right)^{\delta} d\mu\right] \\ &\geq \frac{1}{\delta n}\log\left[\int \mu\left(\mathcal{B}_n(x,\varepsilon)\right)^{-\delta/2} \mu\left(\mathcal{B}_n(x,\varepsilon)\right)^{\delta/2} d\mu\right]^2 = 0 \end{aligned}$$

by the Cauchy–Schwartz inequality. Hence, $I(q',\varepsilon,n) \geq I(q'',\varepsilon,n)$. Finally, by the choice of $\delta$ we have $q_1 \leq q' < 1$ and $1 < q'' \leq q_2$. Since we have already established (2) for $q_1,q_2 \in (-\infty,1)$ or $(1,\infty)$, we conclude that

$$I(q_1,\varepsilon,n) \geq I(q',\varepsilon,n) \geq I(q'',\varepsilon,n) \geq I(q_2,\varepsilon,n).$$

This finishes the proof. $\square$

The second statement in the above lemma will be later extended to all $q \in \mathbb{R}$, including $q=1$. But before we will be able to do it, we will have to establish some additional properties of $I(1,\varepsilon,n)$, see Section 2.3 below.

**Definition 2.2.** *We define the lower and upper correlation entropies of order* $q$ *as*

$$\underline{H}_\mu(f,q) := \lim_{\varepsilon\to 0}\liminf_{n\to\infty} I(q,\varepsilon,n), \quad \overline{H}_\mu(f,q) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty} I(q,\varepsilon,n).$$

*If* $\underline{H}_\mu(f,q) = \overline{H}_\mu(f,q)$ *for some* $q \in \mathbb{R}$, *then we say that* $H_\mu(f,q)$ *— the correlation entropy of order* $q$, *exists and put* $H_\mu(f,q) := \underline{H}_\mu(f,q) = \overline{H}_\mu(f,q)$.

Note that the limit $\varepsilon \to 0$ exists due to the result of Lemma 2.1. Also, the monotonicity of $I(q,\varepsilon,n)$ with respect to $q$ will be preserved in the limit, and hence

$$\underline{H}_\mu(f,q_1) \geq \underline{H}_\mu(f,q_2), \quad \overline{H}_\mu(f,q_1) \geq \overline{H}_\mu(f,q_2)$$

for all $q_1<q_2$, $q_1,q_2 \neq 1$.

## 2.2 Measure-theoretic and topological entropies

In this section we recall the definitions and state without the proofs the results, which we shall use later. We start with the definition of measure–theoretic (or Kolmogorov–Sinai) entropy [8]. We then give a definition of topological entropy, formulate the variational principle and recall two important results due to A. Katok [5] and M. Brin, A. Katok [4].

### 2.2.1 Measure-theoretic entropy

Let $(X, \mathfrak{B}, \mu)$ be a probability space, and $f : X \to X$ be a measure–preserving transformation, i.e., for every $A \in \mathfrak{B}$, $f^{-1}A \in \mathfrak{B}$ and

$$\mu(A) = \mu(f^{-1}A).$$

The dynamical system $(X, \mathfrak{B}, \mu, f)$ is called *ergodic* if every invariant set has measure 0 or 1:

$$\forall A \in \mathfrak{B} : \ A = f^{-1}A \ \Rightarrow \ \mu(A) = 0 \ \text{ or } \ \mu(A) = 1.$$

**Definition 2.3.** *Let $\xi$ be a finite or countable partition of $X$ into measurable sets $\{\Delta_i\}$. The entropy of the partition $\xi$ is defined as the value*

$$H(\xi) = -\sum_i \mu(\Delta_i) \log \mu(\Delta_i).$$

*For all uncountable partitions: $H(\xi) = \infty$.*

Denote by $\xi^{(k)}$ the partition of $X$ into all non-empty sets of the form

$$\Delta_{i_0,\dots,i_{k-1}} := \Delta_{i_0} \cap f^{-1}\Delta_{i_1} \cap \dots \cap f^{-k+1}\Delta_{i_{k-1}}.$$

The partition $\xi^{(k)}$ is called the $k$-th iterate of $\xi$. One can show that for any at most countable partition $\xi$ with $H(\xi) < \infty$, the limit

$$h_\mu(f, \xi) := \lim_{k\to\infty} \frac{1}{k} H(\xi^{(k)})$$

exists and is finite. This limit is called the *measure–theoretic entropy of $f$ with respect to $\xi$.*

**Definition 2.4.** *The measure–theoretic entropy of $f$ is defined as*

$$h_\mu(f) := \sup_{\xi :\, H(\xi)<\infty} h_\mu(f, \xi).$$

The following theorem is due to Shannon, McMillan and Breiman, see [8].

**Theorem 2.5.** *For any $x \in X$, denote by $\xi^{(k)}(x)$ the unique element of the partition $\xi^{(k)}$ containing $x$. Assume $(X, \mathfrak{B}, \mu, f)$ is an ergodic dynamical system. Then for $\mu$-almost all $x \in X$:*

$$\lim_{k\to\infty} -\frac{1}{k} \log \mu(\xi^{(k)}(x)) = h_\mu(f, \xi).$$

This result shows that most elements of $\xi^{(k)}$ have approximately the same measure $\exp(-kh_\mu(f,\xi))$. Using the fact that almost sure convergence implies the convergence in probability, the previous statement can be rewritten in the following form:

**Corollary 2.6 (Equipartition property of ergodic dynamical systems).** *Assume that $(X, \mathfrak{B}, \mu, f)$ is an ergodic dynamical system. Then for any partition $\xi$ and for each $\delta > 0$ there exists $K = K(\xi, \delta)$ such that for any $k > K$ one can choose a $\xi^{(k)}$–measurable set $G_k$ (i.e., $G_k$ is a union of some elements of $\xi^{(k)}$) such that*

*1) $\mu(G_k) \geq 1 - \delta$;*

*2) if $x \in G_k$, then $\exp\bigl(-(h_\mu(f,\xi)+\delta)k\bigr) \leq \mu(\xi^{(k)}(x)) \leq \exp\bigl(-(h_\mu(f,\xi)-\delta)k\bigr)$.*

### 2.2.2 Topological entropy

We now give a definition of the topological entropy. This definition has been proposed by Bowen, the original definition was given by Adler, Konheim and McAndrew. Bowen's definiton is somewhat easier to use in practical computations.

Consider a compact metric space $(X, d)$ together with a continuous transformation $f : X \to X$. We say that a subset $S \in X$ is $(n, \varepsilon)$–generating if for every $x \in X$ there exists $y \in S$ such that

$$d_n(x,y) = \max_{i=0,\dots,n-1} d(f^i x, f^i y) < \varepsilon.$$

Let $N(n, \varepsilon)$ be the least number of points in an $(n, \varepsilon)$–generating set. Then the following limit exists and is called the *topological entropy of $f$*

$$h_{top}(f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon).$$

One can also show that

$$h_{top}(f) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon).$$

Another equivalent approach consists of considering $(n, \varepsilon)$*-separated sets.* A set $E$ is called $(n, \varepsilon)$–separated if for every $x, y \in E$ there exists $0 \leq i < n$ such that $d(f^i x, f^i y) > \varepsilon$, i.e., $d_n(x, y) > \varepsilon$. Let $S(n, \varepsilon)$ be the maximal cardinality of an $(n, \varepsilon)$–separated set. Then

$$h_{top}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S(n,\varepsilon) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log S(n,\varepsilon).$$

The fundamental relation between the topological and measure-theoretic entropies is given by the following theorem.

**Theorem 2.7 (Variational principle).** *Let $(X,d)$ be a compact metric space and $f : X \to X$ a continuous map. Then the topological entropy of $f$ satisfies*

$$h_{top}(f) = \sup_{\mu \in \mathcal{M}_f(X)} h_\mu(f),$$

*where $\mathcal{M}_f(X)$ is the set of all $f$–invariant Borel probability measures on $X$.*

This result leads to the following definition.

**Definition 2.8.** *If $h_{top}(f) = h_\mu(f)$ for some $\mu \in \mathcal{M}_f(X)$, then $\mu$ is called a measure of maximal entropy.*

A. Katok in [5] showed that one can define the measure-theoretic entropy in terms, similar to the definition of the topological entropy.

**Theorem 2.9.** *Let $(X,d)$ be a compact metric space and $f : X \to X$ a continuous map. Assume, $\mu$ is an ergodic $f$–invariant Borel probability measure on $X$. Then for every $\delta \in (0,1)$*

$$h_\mu(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta),$$

*where $N(n,\varepsilon,\delta)$ is the least number of $\varepsilon$–balls in the $d_n$–metric which cover a set of measure at least $1-\delta$.*

The following theorem by M. Brin and A. Katok [4] may be considered as a topological version of the Shannon–McMillan–Breiman theorem.

**Theorem 2.10.** *Let $(X,d)$ be a compact metric space and $f : X \to X$ be a continuous map preserving a non–atomic Borel probability measure $\mu$. Then for $\mu$-almost all $x \in X$*

*1)* $\displaystyle \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu\big(\mathcal{B}_n(x,\varepsilon)\big) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu\big(\mathcal{B}_n(x,\varepsilon)\big) = h_\mu(f,x);$

*2)* *$h_\mu(f,x)$ is an $f$–invariant function;*

*3)* $\displaystyle \int_X h_\mu(f,x)\, d\mu = h_\mu(f).$

Since for ergodic dynamical systems invariant functions are constants almost everywhere, we have the following corollary.

**Corollary 2.11.** *If $(X,\mu,f)$ is an ergodic dynamical system, then $h_\mu(f,x) = h_\mu(f)$ for $\mu$–almost every $x$.*

## 2.3 Correlation entropy of order 1

In this section we study the properties of $I(1,\varepsilon,n)$ and extend the result of Lemma 2.1 to all $q \in \mathbb{R}$. We also show that the correlation entropy of order 1 exists and is equal to the measure–theoretic entropy. We remind, that we assume that $f$ is a continuous transformation of a compact metric space $(X,d)$, and $\mu$ is an invariant Borel probability measure.

**Lemma 2.12.** *For every $\varepsilon > 0$ and $n \in \mathbb{N}$*

$$I(1,\varepsilon,n) = -\frac{1}{n}\int \log \mu(\mathcal{B}_n(x,\varepsilon))\, d\mu < \infty.$$

*Proof.* Let $\xi$ be an arbitrary finite measurable partition of $X$ such that

$$\operatorname{diam}(\xi) := \max_{\Delta\in\xi}\ \operatorname{diam}(\Delta) < \varepsilon. \tag{2.3}$$

Such partition exists due to compactness of $X$. Then for every $x \in X$ one has $\xi(x) \subseteq \mathcal{B}(x,\varepsilon)$, and hence

$$\xi^{(n)}(x) = \bigcap_{i=0}^{n-1} f^{-i}\xi(f^i x) \subseteq \bigcap_{i=0}^{n-1} f^{-i}\mathcal{B}(f^i x,\varepsilon) = \mathcal{B}_n(x,\varepsilon).$$

Therefore

$$\begin{aligned} I(1,\varepsilon,n) &= -\frac{1}{n}\int \log \mu(\mathcal{B}_n(x,\varepsilon))\, d\mu \le -\frac{1}{n}\int \log \mu(\xi^{(n)}(x))\, d\mu \\ &= \frac{1}{n}\left(-\sum_{\Delta\in\xi^{(n)}} \mu(\Delta)\log\mu(\Delta)\right) \le \frac{1}{n}\log \operatorname{card}(\xi^{(n)}) \le \log\operatorname{card}(\xi), \end{aligned}$$

where we have used the Shannon inequality:

$$-\sum_{i=1}^{K} p_i \log p_i \le \log K \quad \text{for every } (p_1,\dots,p_K):\ p_i \ge 0,\ \sum_{i=1}^{K} p_i = 1,$$

and the fact that $\operatorname{card}(\xi^{(n)}) \le \big(\operatorname{card}(\xi)\big)^n$. Thus, $I(1,\varepsilon,n)$ is finite for all $\varepsilon > 0$ and $n \in \mathbb{N}$. □

Now we can add the missing case of $q = 1$ to the statement of Lemma 2.1.

**Lemma 2.13.** *For all $q_1 < 1$ and $q_2 > 1$*

$$I(q_2,\varepsilon,n) \le I(1,\varepsilon,n) \le I(q_1,\varepsilon,n).$$

*Proof.* First we recall the Jensen inequality: for a convex function $g(t) : \mathbb{R} \to \mathbb{R}$, and $h : X \to \mathbb{R}$ such that

$$\int |h(x)|\, d\mu < \infty,$$

the following inequality holds

$$g\Big(\int h(x)d\mu\Big) \le \int g(h(x))\, d\mu.$$

Let $q \in \mathbb{R}$ and $q \ne 1$. Take $g(t) = \exp((q-1)t)$ and $h(x) = \log \mu(\mathcal{B}_n(x,\varepsilon))$. Since $g(t)$ is convex, and in the previous lemma we have established that

$$\int |\log \mu(\mathcal{B}_n(x,\varepsilon))| d\mu < \infty,$$

we can apply the Jensen inequality to $g$ and $h$, and after taking logarithm of both sides of the inequality, we finally obtain

$$(q-1)\int \log \mu(\mathcal{B}_n(x,\varepsilon))d\mu \le \log \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1} d\mu \tag{2.4}$$

for all $q \in \mathbb{R}$, $q \ne 1$. Now the statement of lemma follows easily. $\square$

Using the Brin-Katok formula we can actually compute $\underline{H}_\mu(f,1)$ and $\overline{H}_\mu(f,1)$.

**Lemma 2.14.** *The correlation entropy of order* 1 *exists, and*

$$H_\mu(f,1) = \underline{H}_\mu(f,1) = \overline{H}_\mu(f,1) = h_\mu(f).$$

*Hence, for all* $q_1 < 1$ *and* $q_2 > 1$*, we have*

$$\begin{aligned} \overline{H}_\mu(f,q_1) &\ge \underline{H}_\mu(f,q_1) \ge h_\mu(f), \\ \underline{H}_\mu(f,q_2) &\le \overline{H}_\mu(f,q_2) \le h_\mu(f). \end{aligned}$$

*Proof.* Let us start by showing that $\overline{H}_\mu(f,1) \le h_\mu(f)$. Fix $\varepsilon > 0$ and let $\xi$ be any finite partition of $X$ with $\operatorname{diam}(\xi) < \varepsilon$. As in the proof of Lemma 2.12, for every $n \ge 1$ we get that

$$I(1,\varepsilon,n) = -\frac{1}{n}\int \log \mu(\mathcal{B}_n(x,\varepsilon))\, d\mu \le \frac{1}{n} H(\xi^{(n)}),$$

and hence

$$\limsup_{n\to\infty} I(1,\varepsilon,n) \le \limsup_{n\to\infty} \frac{1}{n} H(\xi^{(n)}) = h_\mu(f,\xi) \le h_\mu(f).$$

Thus after taking the corresponding limit $\varepsilon \to 0$, we conclude that $\overline{H}_\mu(f,1) \le h_\mu(f)$.

To estimate $\underline{H}_\mu(f,1)$ from below we will use the Fatou lemma and the Brin-Katok formula. From the Fatou lemma we obtain

$$\int \Big(\lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x,\varepsilon))\Big) d\mu \le \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n}\int \log \mu(\mathcal{B}_n(x,\varepsilon)) d\mu.$$

But according to the Brin-Katok formula

$$h_\mu(f) = \int \underline{h}_\mu(f,x)\, d\mu = \int \Big( \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x,\varepsilon)) \Big)\, d\mu.$$

Hence, $h_\mu(f) \le \underline{H}_\mu(f,1)$ and since we have already established the inequality $\overline{H}_\mu(f,1) \le h_\mu(f)$, we conclude that $H_\mu(f,1)$ exists and

$$H_\mu(f,1) = \underline{H}_\mu(f,1) = \overline{H}_\mu(f,1) = h_\mu(f).$$

$\square$

## 2.4 Correlation entropies of order $q < 1$

In this section we will discuss upper estimates for the correlation entropies of order $q < 1$. The main question which we will try to answer in this section, is whether the correlation entropies are finite or infinite for $q < 0$.

Let us start we a simple estimate. Suppose as before that $(X,d)$ is a compact metric space, $f : X \to X$ is a continuous transformation, $\mu$ is an invariant measure, positive on open sets, and $E = \{x_i\}_i$ is a finite $(n,\varepsilon)$-generating set, i.e., for every $x \in X$, there exists $i$ such that $d_n(x,x_i) < \varepsilon$. Then for $q < 1$ one has

$$\begin{aligned} I(q,2\varepsilon,n) &= \frac{1}{(1-q)n} \log \int \frac{1}{\mu(\mathcal{B}_n(x,2\varepsilon))^{1-q}}\, d\mu \\ &\le \frac{1}{(1-q)n} \log \sum_{x_i\in E} \int\limits_{\mathcal{B}_n(x_i,\varepsilon)} \frac{1}{\mu(\mathcal{B}_n(x,2\varepsilon))^{1-q}}\, d\mu \\ &\le \frac{1}{(1-q)n} \log \sum_{x_i\in E} \mu(\mathcal{B}_n(x_i,\varepsilon))^q, \end{aligned} \tag{2.5}$$

since $\mathcal{B}_n(x_i,\varepsilon) \subseteq \mathcal{B}_n(x,2\varepsilon)$ for every $x \in \mathcal{B}_n(x_i,\varepsilon)$. Hence, for $q = 0$ we have

$$I(0,2\varepsilon,n) \le \frac{1}{n} \log N(n,\varepsilon),$$

where $N(n,\varepsilon)$ is the least cardinality of an $(n,\varepsilon)$-generating set. Thus

$$\overline{H}_\mu(f,0) \le h_{top}(f),$$

and the same estimate is valid for all $q \in (0,1)$ due to the monotonicity of $\overline{H}_\mu(f,q)$ with respect to $q$. Combining this estimate with the result of the previous section we obtain the following statement.

**Theorem 2.15.** *If $\mu$ is an invariant measure with $supp(\mu) = X$, then*

$$h_\mu(f) \le \underline{H}_\mu(f,q) \le \overline{H}_\mu(f,q) \le h_{top}(f)$$

*for all $q \in [0,1)$.*

Now, we consider the case $q<0$. The main result of this section is the following theorem.

**Theorem 2.16.** *Suppose $\mu$ is an invariant measure with supp$(\mu)=X$. Assume also that $\mu$ satisfies the* **weak entropy-doubling condition**, *i.e. for all sufficiently small $\varepsilon>0$ one has*

$$\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X}\frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2))}=0.$$

*Then*

$$\overline{H}_\mu(f,q)<\infty \ \textit{ for some } q<0 \ \Rightarrow\ \overline{H}_\mu(f,q)<\infty \ \textit{ for all } q<0.$$

*Proof.* Suppose $\overline{H}_\mu(f,q_0)<\infty$ for some $q_0<0$. Then $I(q_0,\varepsilon/2,n)<\infty$ for all $\varepsilon>0$ and $n\in\mathbb{N}$. Hence for all $y\in X$ one has

$$\begin{aligned} I(q_0,\varepsilon/2,n) &= \frac{1}{(1+|q_0|)n}\log\int\frac{d\mu}{\mu(\mathcal{B}_n(x,\varepsilon/2))^{1+|q_0|}}\\ &\geq \frac{1}{(1+|q_0|)n}\log\int\limits_{\mathcal{B}_n(y,\varepsilon/2)}\frac{d\mu}{\mu(\mathcal{B}_n(x,\varepsilon/2))^{1+|q_0|}}\\ &\geq \frac{1}{(1+|q_0|)n}\log\frac{\mu(\mathcal{B}_n(y,\varepsilon/2))}{\mu(\mathcal{B}_n(y,\varepsilon))^{1+|q_0|}}, \end{aligned}\tag{2.6}$$

since $\mathcal{B}_n(x,\varepsilon/2)\subseteq\mathcal{B}_n(y,\varepsilon)$ for all $x\in\mathcal{B}_n(y,\varepsilon/2)$. Hence, for every $y\in X$ we have

$$\frac{1}{\mu(\mathcal{B}_n(y,\varepsilon))}\leq\exp\left(\frac{1+|q_0|}{|q_0|}nI(q_0,\varepsilon/2,n)+\frac{1}{|q_0|}F(\varepsilon,n)\right),\tag{2.7}$$

where

$$F(\varepsilon,n)=\log\sup_{x\in X}\frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2))}.$$

We have already established above (see (2.5)), that for any $q<0$

$$I(q,2\varepsilon,n)\leq\frac{1}{(1+|q|)n}\log\sum_{x_j\in E}\frac{1}{\mu(\mathcal{B}_n(x_j,\varepsilon))^{|q|}},$$

where $E$ is an arbitrary $(n,\varepsilon)$-generating set. Combining the previous inequality and (2.7), for a minimal $(n,\varepsilon)$-generating set $E$ we obtain

$$I(q,2\varepsilon,n)\leq\frac{1}{1+|q|}\left[\frac{1}{n}\log N(\varepsilon,n)+(1+|q_0|)\frac{|q|}{|q_0|}I\Big(q_0,\frac{\varepsilon}{2},n\Big)+\frac{|q|}{|q_0|}\frac{F(\varepsilon,n)}{n}\right],$$

where $N(\varepsilon,n)$ is the cardinality of a minimal $(n,\varepsilon)$-generating set. Hence

$$\overline{H}_\mu(f,q)\leq\frac{1}{1+|q|}h_{top}(f)+\frac{1+|q_0|}{1+|q|}\frac{|q|}{|q_0|}\overline{H}_\mu(f,q_0).$$

To complete the proof, we must establish that $h_{top}(f) < \infty$. Firstly, note that due to monotonicity with respect to $q$ we have

$$\overline{H}_\mu(f,0) \le \overline{H}_\mu(f,q_0) < \infty.$$

Let $S$ be a maximal $(n,\varepsilon)$-separated set, and $M(\varepsilon,n)$ its cardinality. Then, in complete analogy to (2.6), we obtain

$$\begin{aligned} I(0,\varepsilon/2,n) &= \frac{1}{n}\log\int \frac{d\mu}{\mu(\mathcal{B}_n(x,\varepsilon/2))} \\ &\ge \frac{1}{n}\log\sum_{x_i\in S}\int_{\mathcal{B}_n(x_i,\varepsilon/2)} \frac{d\mu}{\mu(\mathcal{B}_n(x,\varepsilon/2))} \\ &\ge \frac{1}{n}\log\sum_{x_i\in S}\frac{\mu(\mathcal{B}_n(x_i,\varepsilon/2))}{\mu(\mathcal{B}_n(x_i,\varepsilon))}, \end{aligned} \tag{2.8}$$

and hence

$$\frac{1}{n}\log M(\varepsilon,n) \le I(0,\varepsilon/2,n) + \frac{1}{n}F(\varepsilon,n),$$

and thus, $h_{top}(f) \le \underline{H}_\mu(f,0) < \infty$. This finishes the proof. □

Combining the fact that $h_{top}(f) \le \underline{H}_\mu(f,0)$ for measures with weak entropy-doubling condition, with the result of Theorem 2.15 we obtain the following corollary.

**Corollary 2.17.** *Suppose $\mu$ is an invariant measure with $supp(\mu) = X$, satisfying the weak entropy-doubling condition. Then the correlation entropy of order $0$ exists and*

$$H_\mu(f,0) = \underline{H}_\mu(f,0) = \overline{H}_\mu(f,0) = h_{top}(f).$$

If we introduce the quantity

$$q_* = \inf\{q \in \mathbb{R} : \ \overline{H}_\mu(f,q) < \infty\},$$

then the result of Theorem 2.16 reads: $q_*$ is either $0$ or $-\infty$, provided $\mu$ satisfies the weak entropy doubling condition. In the next section we give a few examples of measures with $q_* = -\infty$, and in the last section of this chapter we give an example of an invariant measure with $q_* = 0$. At the present moment we are not aware of any example for which $q_*$ takes a finite, but negative value.

## 2.5 Examples

In this section we consider several examples of dynamical systems and invariant measures, for which we are able to compute the correlation entropies of all orders explicitly.

### 2.5.1 Homogeneous measures

For a general definition of homogeneous measures see [9], here we only consider the case of compact metric spaces.

**Definition 2.18.** *Let $f$ be a continuous transformation of a compact metric space $(X,d)$. A Borel probability measure $\mu$ on $X$ is said to be $f$–homogeneous if for each $\varepsilon > 0$ there exist $\delta > 0$ and $c > 0$ such that*

$$\mu(\mathcal{B}_n(y,\delta)) \le c\mu(\mathcal{B}_n(x,\varepsilon))$$

*for all $n \in \mathbb{N}$ and $x, y \in X$.*

Homogeneous measures have been introduced by Bowen in [2]. He observed that on certain groups (so-called *homogeneous spaces*) Haar measures, invariant under affine transformations, have such property.

Maybe the simplest example of a homogeneous measure is the Lebesgue measure, invariant under the Arnold-Thom cat map, i.e., a linear automorphism of the two-dimensional torus $\mathbb{T}^2$, given by a matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The Bowen-Margulis measures for Anosov systems are homogeneous as well, see [6].

This homogeneity of a measure is a rather strong property. As the next result shows, homogeneous measures are measures of maximal entropy. The local entropies, defined via the Brin-Katok formula, exist at every point, and the corresponding convergence is, in fact, uniform. Moreover, as we will see later, homogeneous measures have a trivial (degenerate) multifractal spectrum for local entropies.

**Theorem 2.19 ([9]).** *Let $\mu$ be an invariant $f$–homogeneous measure. Then for every $x \in X$, $h_\mu(f,x)$ exists and*

$$h_\mu(f,x) = h_\mu(f) = h_{top}(f).$$

Let us compute the correlation entropies of homogeneous measures. Assume, $\mu$ is an $f$–invariant homogeneous measure. Take an arbitrary $\varepsilon > 0$ and choose the corresponding $\delta = \delta(\varepsilon) > 0$ and $c = c(\varepsilon) > 0$ as in Definition 2.18. Without loss of generality we may assume that $\delta(\varepsilon) \le \varepsilon$, and hence $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. For $q > 1$ one has

$$I(q,\delta,n) = -\frac{1}{(q-1)n} \log \int \mu(\mathcal{B}_n(y,\delta))^{q-1}\, d\mu \ge -\frac{1}{n} \log c\mu(\mathcal{B}_n(x,\varepsilon))$$

for all $x \in X$. Hence

$$\begin{aligned} \underline{H}_\mu(f,q) = \lim_{\delta\to 0} \liminf_{n\to\infty} I(q,\delta,n) &\ge \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log c\mu(\mathcal{B}_n(x,\varepsilon)) \\ &= h_\mu(f,x) = h_\mu(f). \end{aligned}$$

However, since we have already established that $\underline{H}_\mu(f,q) \le \overline{H}_\mu(f,q) \le h_\mu(f)$ for all $q > 1$, we finally obtain

$$\underline{H}_\mu(f,q) = \overline{H}_\mu(f,q) = h_\mu(f) \quad \text{for all} \quad q > 1.$$

The remaining case $q < 1$ can be treated similarly. Hence, we proved the following statement.

**Theorem 2.20.** *Let $\mu$ be an invariant $f$–homogeneous measure on $X$. Then for any $q \in \mathbb{R}$ one has*

$$H_\mu(f,q) = \underline{H}_\mu(f,q) = \overline{H}_\mu(f,q) = h_\mu(f) = h_{top}(f).$$

### 2.5.2 Bernoulli measures

Let $\Omega = \{1,\dots,m\}$ be a finite alphabet, and let $X = \Omega^{\mathbb{Z}} = \{\mathbf{x} = \{x_i\}_{i=-\infty}^{\infty} : x_i \in \Omega\}$ be the set of all doubly infinite sequences. We denote by $\sigma$ the left shift on $X$:

$$(\sigma\mathbf{x})_i = x_{i+1}.$$

We equip $X$ with a standard metric

$$d(\mathbf{x},\mathbf{y}) = 2^{-N},$$

where $N = \max\{n \in \mathbb{N} : \ x_i = y_i \ \text{ for all } i \text{ with } |i| < n\}$; if $x_0 \neq y_0$, then $N = 0$.

For any $s,t$, $s \leq t$, and any set $\{a_s,\dots,a_t\}$, $a_i \in \Omega$, we define a cylinder

$$C_s^t(a_s,\dots,a_t) = \Big\{\mathbf{x} \in X : \ x_i = a_i \text{ for } i = s,\dots,t\Big\}.$$

For the symbolic space $X$ and the metric $d$ as above, we can approximate sets $\mathcal{B}_n(\mathbf{x},\varepsilon)$ by cylinders. Namely, let $\mathbf{x}$ be an arbitrary point of $X$ and take $\varepsilon = 1/2^{k+1}$, $k \in \mathbb{N}$. Then for every $\mathbf{y} \in \mathcal{B}_n(\mathbf{x}, 2^{-(k+1)})$, we have

$$x_i = y_i \ \text{ for all } \ i = -k,\dots,k+n-1,$$

and hence

$$\mathcal{B}_n(x, 2^{-(k+1)}) \subseteq C_{-k}^{k+n-1}(x_{-k},\dots,x_{k+n-1}). \tag{2.9}$$

Similarly

$$C_{-k-1}^{k+n-1}(x_{-k-1},\dots,x_{n+k-1}) \subseteq \mathcal{B}_n(x, 2^{-(k+1)}). \tag{2.10}$$

Let us introduce the following notation: $\displaystyle\sum_{C\in(s,t)}$ will denote the summation over all possible cylinders $C$ starting at $s$ and ending at $t$.

Suppose $\mu$ is some shift invariant measure, which is positive on open subsets of $X$. Then using (2.9) and (2.10) we can estimate $\int \mu(\mathcal{B}_n(\mathbf{x}, 2^{-(k+1)})^{q-1} d\mu$ as follows: for $q < 1$

$$\sum_{C'\in(-k,k+n-1)} \mu\left(C'\right)^q \leq \int \mu\big(\mathcal{B}_n(\mathbf{x}, 2^{-(k+1)})\big)^{q-1} d\mu \leq \sum_{C''\in(-k-1,k+n-1)} \mu\left(C''\right)^q, \tag{2.11}$$

and for $q > 1$

$$\sum_{C'' \in (-k-1,k+n-1)} \mu(C'')^q \le \int \mu\big(\mathcal{B}_n(\mathbf{x}, 2^{-(k+1)})\big)^{q-1} d\mu \le \sum_{C' \in (-k,k+n-1)} \mu(C')^q . \tag{2.12}$$

Note that for $q = 0$ estimates in (2.11) do not actually depend on $\mu$. The integral $I(0, 2^{-(k+1)}, n)$ is estimated from above (below) by the logarithm of a number of different cylinders starting at $-(k+1)$ $(-k)$ respectively, and ending at $k+n-1$. Hence, for $q = 0$ we have

$$\log m^{2k+n} \le I(0, 2^{-(k+1)}, n) \le \log m^{2k+n+1},$$

and therefore

$$\underline{H}_\mu(\sigma, 0) = \overline{H}_\mu(\sigma, 0) = \log m.$$

Finally, we recall that $h_{top}(\sigma) = \log m$, see [6], and hence we have proved the following statement.

**Lemma 2.21.** *Let $\mu$ be a shift invariant measure on $X$, such that $supp(\mu) = X$. Then the correlation entropy of order $0$ exists and*

$$H_\mu(f, 0) = h_{top}(\sigma).$$

Next we compute the correlation entropies for Bernoulli measures. Let $\mathbf{p} = (p_1, \dots, p_m)$ be a probability vector, i.e $p_i \ge 0$ for all $i$ and $\sum_i p_i = 1$. Define a measure $\rho = \rho(\mathbf{p})$ on $\Omega$ by letting $\rho(\{i\}) = p_i$ for every $i = 1, \dots, m$. Let $\mu = \rho^{\mathbb{Z}}$ be a corresponding product measure on $X = \Omega^{\mathbb{Z}}$. Then for any $s, t$, $s \le t$, and any set $\{a_s, \dots, a_t\} \in \Omega$, the measure of a corresponding cylinder is given by

$$\mu(C_s^t(a_s, \dots, a_t)) = \prod_{i=s}^{t} p_{a_i}.$$

The measure $\mu$ is called the Bernoulli measure on $X$. For this measure the measure–theoretic entropy is $h_\mu(\sigma) = -(p_1 \log p_1 + \dots + p_m \log p_m)$. Estimates (2.11), (2.12) now take the following form:

$$q < 1: \quad (p_1^q + \dots + p_m^q)^{2k+n} \le \int \mu\big(\mathcal{B}_n(\mathbf{x}, 2^{-(k+1)})\big)^{q-1} d\mu \le (p_1^q + \dots + p_m^q)^{2k+n+1},$$

$$q > 1: \quad (p_1^q + \dots + p_m^q)^{2k+n+1} \le \int \mu\big(\mathcal{B}_n(\mathbf{x}, 2^{-(k+1)})\big)^{q-1} d\mu \le (p_1^q + \dots + p_m^q)^{2k+n}.$$

Taking all the necessary limits, we finally obtain that the correlation entropy of order $q$ exists for all $q \in \mathbb{R}$, $q \ne 1$, and

$$H_\mu(\sigma, q) = \underline{H}_\mu(\sigma, q) = \overline{H}_\mu(\sigma, q) = -\frac{1}{q-1} \log(p_1^q + \dots + p_m^q).$$

Obviuosly, $H_\mu(\sigma,q)$ depends continuously on $q$ for $q<1$ and $q>1$. Taking the limit $q\to 1$ in the above expression, we see that

$$\begin{aligned}\lim_{q\to 1} H_\mu(\sigma,q) &= \lim_{q\to 1} -\frac{1}{q-1}\log(p_1^q+\ldots+p_m^q) = \lim_{q\to 1} -\frac{p_1^q\log p_1+\ldots+p_m^q\log p_m}{p_1^q+\ldots+p_m^q}\\ &= -(p_1\log p_1+\ldots+p_m\log p_m) = h_\mu(\sigma) = H_\mu(\sigma,1).\end{aligned}$$

Hence, $H_\mu(\sigma,q)$ is a continuous function of $q$ on the whole $\mathbb{R}$. Moreover, one can actually show that $H_\mu(\sigma,q)$ is a real-analytic function of $q$, even at $q=1$.

### 2.5.3 Gibbs measures

In the previous section we have been able to compute the correlation entropies of all orders for Bernoulli measures. The key to our success was the fact, that we had a good knowledge about the measure of each cylinder. There is a wider class of measures, for which one has a good control over the measures of cylindric sets. These measures — the so-called Gibbs measures, are the subject of this section. In our presentation of Gibbs measures we follow [3].

As in the previous section let $\Omega=\{1,\ldots,m\}$, and $X=\Omega^{\mathbb{Z}}$, $\sigma$ denotes the left shift. For a continuous function $\varphi: X\to\mathbb{R}$ define

$$\mathrm{var}_n(\varphi) := \sup\{|\varphi(\mathbf{x})-\varphi(\mathbf{y})| :\ x_i=y_i \ \text{ for all } i \ \text{ with } |i|<n\}.$$

For each $\varphi$ with sufficiently fast decaying $\mathrm{var}_n$, one is able to construct a special measure, called the Gibbs measure.

**Theorem 2.22.** *Suppose that $\varphi: X\to\mathbb{R}$ is such that*

$$\mathrm{var}_n(\varphi)\le c\alpha^n \tag{2.13}$$

*for some $c>0$ and $\alpha\in(0,1)$. Then there exists a unique shift-invariant Borel probability measure $\mu=\mu_\varphi$ such that one can find constants $c_1,c_2>0$ and $P$ satisfying*

$$c_1 \le \frac{\mu\big(\{\mathbf{y}:\ y_i=x_i\ \ \forall i=0,\ldots,n-1\}\big)}{\exp\big(-Pn+\sum_{i=0}^{n-1}\varphi(\sigma^i\mathbf{x})\big)} \le c_2. \tag{2.14}$$

*for all $\mathbf{x}\in X$ and $n\in\mathbb{N}$.*

This measure $\mu=\mu_\varphi$ is called the Gibbs measure corresponding to (a potential) $\varphi$. The constant $P=P(\varphi)$, which appears in (2.14), is called the (topological) pressure of $\varphi$. In fact, there is another definition of the topological pressure, which is suitable for all continuous functions, and not only those, satisfying (2.13).

**Definition 2.23.** *For each $n$-tuple $(a_0,\ldots,a_{n-1})$, $a_i\in\Omega$, let*

$$\sup_{a_0,\ldots,a_{n-1}} S_n(\varphi) := \sup\Big\{\sum_{i=0}^{n-1}\varphi(\sigma^i\mathbf{x}) :\ x_i=a_i \ \ \textit{for}\ \ 0\le i<n-1\Big\},$$

*and*

$$Z_n(\varphi) := \sum_{a_0,\dots,a_{n-1}} \exp\Big(\sup_{a_0,\dots,a_{n-1}} S_n(\varphi)\Big).$$

*Then the following limit exists*

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi)$$

*and is called the topological pressure of* $\varphi$.

**Theorem 2.24 (Variational Principle).** *Suppose* $\varphi : X \to \mathbb{R}$ *is a continuous function and* $\mu$ *is an invariant measure, then*

$$h_\mu(\sigma) + \int \varphi\, d\mu \leq P(\varphi).$$

*If* $\varphi$ *satisfies* (2.13), *then the corresponding measure* $\mu_\varphi$ *from Theorem 2.22 is the unique invariant measure such that*

$$h_\mu(\sigma) + \int \varphi\, d\mu = P(\varphi).$$

The next result expresses the correlation entropies of Gibbs measures in terms of topological pressure.

**Theorem 2.25.** *Let* $\varphi : X \to \mathbb{R}$ *be a continuous function satisfying* (2.13), *and let* $\mu$ *be the corresponding Gibbs measure. Then the correlation entropy of order* $q$ *exists for every* $q \in \mathbb{R}$, *and for each* $q \neq 1$ *we have*

$$\begin{aligned} H_\mu(\sigma, q) = \underline{H}_\mu(\sigma, q) = \overline{H}_\mu(\sigma, q) &= \frac{P(q\varphi) - qP(\varphi)}{1-q} \\ &= P(\varphi) - \frac{P(\varphi + (q-1)\varphi) - P(\varphi)}{q-1}. \end{aligned}$$

*Proof.* Using the characteristic property of Gibbs measures (2.14),for $q < 1$ we can continue the estimates (2.11) as follows:

$$\begin{aligned} \int \mu\big(\mathcal{B}_n(\mathbf{x}, 2^{-(k+1)})\big)^{q-1} d\mu &\geq \sum_{a_{-k},\dots,a_{k+n-1}} \mu\big(C_{-k}^{k+n-1}(a_{-k},\dots,a_{k+n-1})\big)^q \\ &= \sum_{a_{-k},\dots,a_{k+n-1}} \mu\big(C_0^{2k+n-1}(a_{-k},\dots,a_{k+n-1})\big)^q \\ &\geq c_1^q \exp\big(-P(2k+n)q\big) Z_{2k+n}(q\varphi), \end{aligned}$$

and similarly

$$\int \mu\big(\mathcal{B}_n(\mathbf{x}, 2^{-(k+1)})\big)^{q-1} d\mu \leq c_2^q \exp\big(-P(2k+n+1)q\big) Z_{2k+n+1}(q\varphi).$$

Hence, for each $q < 1$ we get

$$H_\mu(\sigma, q) = \underline{H}_\mu(\sigma, q) = \overline{H}_\mu(\sigma, q) = \frac{P(q\varphi) - qP(\varphi)}{1-q}.$$

The case $q > 1$ is proved in the same way. $\square$

As in the case of Bernoulli measures, one can show that the correlation entropies $H_\mu(\sigma, q)$, $q \in \mathbb{R}$, depend real-analytically on $q$.

## 2.6 Continuous dependence on the parameter

In this section we prove that the lower $\{\underline{H}_\mu(f,q)\}$ and upper $\{\overline{H}_\mu(f,q)\}$ correlation entropies depend continuously on the parameter $q$ for $q \in (1,\infty)$ and $q \in (q_*, 1)$, where

$$q_* = \inf\Big\{ q \in \mathbb{R} : \ \overline{H}_\mu(f,q) < \infty \Big\}.$$

In the next section we give examples of discontinuous behavior in the family of correlation entropies at $q = 0$ and $q = 1$.

As usual, our running assumptions will be finiteness of the topological entropy of $f$, and the positiveness of $\mu$ on open sets. Let us consider the easier case $q > 1$ first. We start with the following simple estimate: for $1 < q_1 < q_2$ one has

$$I(q_1, \varepsilon, n) \leq \frac{q_2 - 1}{q_1 - 1} I(q_2, \varepsilon, n). \tag{2.15}$$

This inequality follows immediately from the fact that

$$\int \mu(\mathcal{B}_n(x,\varepsilon))^{q_2 - 1} d\mu \leq \int \mu(\mathcal{B}_n(x,\varepsilon))^{q_1 - 1} d\mu$$

for $q_2 > q_1$. Using (2.15) and the monotonicity from Lemma 2.1, for $1 < q_1 < q_2$ we obtain the following inequalities

$$\underline{H}_\mu(f, q_2) \leq \underline{H}_\mu(f, q_1) \leq \frac{q_2 - 1}{q_1 - 1} \underline{H}_\mu(f, q_2) \tag{2.16}$$

$$\overline{H}_\mu(f, q_2) \leq \overline{H}_\mu(f, q_1) \leq \frac{q_2 - 1}{q_1 - 1} \overline{H}_\mu(f, q_2) \tag{2.17}$$

Now, using (2.16) and (2.17), one easily establishes the continuity of $\underline{H}_\mu(f,q)$ and $\overline{H}_\mu(f,q)$ with respect to $q$ for $q > 1$:

**Lemma 2.26.** *The lower and upper correlation entropies $\underline{H}_\mu(f,q)$ and $\overline{H}_\mu(f,q)$ are continuous on $(1, +\infty)$.*

Now we consider the remaining case. For $q_1, q_2 \in (q_*, 1)$, and $\lambda \in (0,1)$, consider

$$q = \lambda q_1 + (1-\lambda) q_2 \ \in (q_*, 1).$$

By the Cauchy-Schwartz inequality one has

$$\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu \leq \Big( \int \mu(\mathcal{B}_n(x,\varepsilon))^{q_1 - 1}\, d\mu \Big)^\lambda \Big( \int \mu(\mathcal{B}_n(x,\varepsilon))^{q_2 - 1}\, d\mu \Big)^{1-\lambda}. \tag{2.18}$$

Consider $\overline{c}(q) = (1-q)\overline{H}_\mu(f,q)$, i.e.,

$$\overline{c}(q) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu.$$

Then from (2.18) we deduce that $\overline{c}(q)$ is convex:

$$\overline{c}(q) \le \lambda\overline{c}(q_1) + (1-\lambda)\overline{c}(q_2)$$

for all $q_1, q_2 \in (q_*, 1)$ and every $\lambda \in (0,1)$. The function $\overline{c}(q)$ is continuous on $(q_*, 1)$ since it is finite and convex. Hence, $\overline{H}_\mu(f,q) = \overline{c}(q)/(1-q)$ is continuous on $(q_*, 1)$ as well.

Now, let us prove continuity of $\underline{H}_\mu(f,q)$. Put

$$J(q,\varepsilon,n) = \frac{1}{n}\log\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu = (1-q)I(q,\varepsilon,n).$$

Suppose $q \in (q_*, 1)$, $t > 0$ and $m \in \mathbb{N}$ are such that $q-(2^m-1)t > q_*$ and $q+t < 1$. Then from (2.18) we have

$$\begin{aligned}
J(q,\varepsilon,n) &\le \frac{1}{2}\Big(J(q-t,\varepsilon,n) + J(q+t,\varepsilon,n)\Big)\\
&= J(q+t,\varepsilon,n) + \frac{1}{2}\Big(J(q-t,\varepsilon,n) - J(q+t,\varepsilon,n)\Big)\\
&\le J(q+t,\varepsilon,n) + \frac{1}{4}\Big(J(q-3t,\varepsilon,n) - J(q+t,\varepsilon,n)\Big)\\
&\dots\\
&\le J(q+t,\varepsilon,n) + \frac{1}{2^m}\Big(J(q-(2^m-1)t,\varepsilon,n) - J(q+t,\varepsilon,n)\Big)\\
&\le J(q+t,\varepsilon,n) + \frac{1}{2^m}J(q-(2^m-1)t,\varepsilon,n).
\end{aligned}$$

Choose arbitrary $\delta > 0$. For a given $q \in (q_*, 1)$, choose some $\gamma > 0$ such that $q_* + \gamma < q$. Since $\overline{H}_\mu(f, q_*+\gamma) < \infty$, for every $\varepsilon > 0$ there exists $N$ such that

$$J(q_*+\gamma,\varepsilon,n) \le (1-q_*-\gamma)\overline{H}_\mu(f,q_*+\gamma) + 1$$

for all $n > N$. Let us choose an integer $m$ and $t > 0$ in a such way that

$$\frac{(1-q_*-\gamma)\overline{H}_\mu(f,q_*+\gamma)+1}{2^m} < \delta, \tag{2.19}$$

$$q-(2^m-1)t > q_*+\gamma, \quad\text{and}\quad q+t<1. \tag{2.20}$$

Then for all $n > N$ one has

$$\begin{aligned}
J(q,\varepsilon,n) &\le J(q+t,\varepsilon,n) + \frac{1}{2^m}J(q-(2^m-1)t,\varepsilon,n)\\
&\le J(q+t,\varepsilon,n) + \frac{1}{2^m}J(q_*+\gamma,\varepsilon,n),
\end{aligned}$$

and hence

$$J(q,\varepsilon,n) \le J(q+t,\varepsilon,n) + \delta.$$

Therefore, for

$$\underline{c}(q) = (1-q)\underline{H}_\mu(f,q) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} J(q,\varepsilon,n),$$

we have

$$\underline{c}(q) \le \underline{c}(q+t) + \delta.$$

Taking into account that $\underline{c}(q)$ is a non-increasing function of $q$, we conclude that for any $q \in (q_*, 1)$ and any $\delta > 0$, we can find $t > 0$ such that for all $q \in (q, q+t)$ one has

$$\left|\underline{c}(q) - \underline{c}(q')\right| < \delta.$$

In a completely similar fashion one can prove that for any $\delta > 0$ there exists $\tilde{t} > 0$ such that for $\tilde{q} \in (q - \tilde{t}, q)$ one has

$$\left|\underline{c}(\tilde{q}) - \underline{c}(q)\right| < \delta.$$

Hence, $\underline{c}(q)$ is continuous on $(q_*, 1)$, and therefore $\underline{H}_\mu(f,q) = \underline{c}(q)/(1-q)$ is continuous as well. Thus we have proved the following result.

**Theorem 2.27.** *The upper and lower correlation entropies of order $q$ are continuous functions of $q$ on $(q_*, 1)$, where*

$$q_* = \inf\bigl\{q \in \mathbb{R} :\ \overline{H}_\mu(f,q) < \infty\bigr\}.$$

## 2.7 Examples of possible singularities at $q = 0$ and $q = 1$

In the previous sections we have seen some examples of invariant measures for which the correlation entropies are real-analytic in $q$. Results of the last section show that there are only two possible points of discontinuity in the family of (upper) correlation entropies. Namely, $q = 1$ and $q_* = \inf\{q \in \mathbb{R} :\ \overline{H}_\mu(f,q) < \infty\}$. In this section we present two examples, which realize each possibility. For the second example $q_* = 0$, and at the present moment we are not aware of any example of an invariant measure with $q_* \in (-\infty, 0)$.

*Example 2.28 (Manneville-Pomeau maps).* For a given $s \in (0,1)$ consider a piecewise continuous and piecewise monotonic transformation of the unit interval $f_s(x)$ given by

$$f_s(x) = x + x^{1+s} \pmod 1.$$

This transformation is known to have a unique ergodic absolutely continuous invariant measure $\mu$ (see the Chapter 7 of this thesis for more details on ergodic properties of $\mu$). Now, we only recall that $\mu$ has positive measure-theoretic entropy $h_\mu(f_s)$ and

$$h_\mu(f_s) = \int \log f_s'\, d\mu.$$

Consider a partition of the unit interval into intervals of monotonicity: $\xi = \{I_0, I_1\}$, where $I_1 = [0, a_s]$, $I_1 = [a_s, 1]$, where $a_s$ is a unique root of the equation

$$a + a^{1+s} = 1.$$

Let us consider the space of all binary sequences

$$X = \{0,1\}^{\mathbb{Z}_+} = \{\omega = (\omega_i) : \omega_i = 0 \text{ or } 1 \text{ for all } i \geq 0\},$$

and let us define a shift invariant probability measure $\nu$ on $X$. Note, that in order to define a measure on $X$ it is sufficient to define only the measures of cylindric sets. We define $\nu$ in the following way

$$\nu\big(C_k^{n+k}(\omega_0, \dots, \omega_n)\big) = \mu(I_{\omega_0} \cap f_s^{-1} I_{\omega_1} \cap \dots \cap f_s^{-n} I_{\omega_n}). \tag{2.21}$$

Clearly, $\nu$ is a shift invariant probability measure on $X$. Note also, that $h_\nu(\sigma) = h_\mu(f_s, \xi) = h_\mu(f_s)$, since $\xi$ is a generating partition.

Finally, some additional analysis (see chapter 7) shows that there exist constants $C > 0$ and $\gamma > 0$ such that

$$\nu(C_0^n(0, 0, \dots, 0)) \geq C n^{-\gamma}$$

for all $n \in \mathbb{N}$. Using the one-sided analogue of the estimate (2.12), for $q > 1$ we obtain

$$\begin{aligned} I(q, 2^{-(k+1)}, n) &= -\frac{1}{(q-1)n} \log \int \nu\big(\mathcal{B}_n(\omega, 2^{-(k+1)})\big)^{q-1} \, d\nu \\ &\leq -\frac{1}{(q-1)n} \log \nu\big(C_0^{n+k}(0, 0, \dots, 0)\big)^q \\ &\leq -\frac{q \log C}{(q-1)n} + \frac{q\gamma}{(q-1)} \frac{\log(n+k)}{n} \to 0 \end{aligned}$$

as $n \to \infty$. Hence

$$\underline{H}_\nu(\sigma, q) = \overline{H}_\nu(\sigma, q) = 0$$

for all $q > 1$. Now, recalling that $\underline{H}_\nu(\sigma, q) \geq h_\nu(\sigma) > 0$ for all $q < 1$, we see that at $q = 1$ the families of generalized entropies $\underline{H}_\nu(\sigma, q)$ and $\overline{H}_\nu(\sigma, q)$ have discontinuities. In fact, using some results of chapter 7, one can show that for every $q \in \mathbb{R}$, $q \neq 1$,

$$\underline{H}_\nu(\sigma, q) = \overline{H}_\nu(\sigma, q) = \frac{P(q\psi_s)}{1-q},$$

where the potential $\psi_s : X \to \mathbb{R}$ is defined as follows: for every infinite sequence $\omega = (\omega_i)_{i \geq 0}$, we define $x = x(\omega) \in [0, 1]$ by

$$x(\omega) = \overline{I_{\omega_0}} \cap f^{-1} \overline{I_{\omega_1}} \cap \dots \cap f^{-n+1} \overline{I_{\omega_n}} \cap \dots .$$

By definition, we put $\psi_s(\omega) = -\log f_s'(x(\omega))$. The pressure function $P(q\psi_s)$ is positive, smooth and strictly convex for $q < 1$ and $P(q\psi_s) \equiv 0$ for all $q > 1$.

The example of the Manneville-Pomeau maps shows that a sub-exponential decay of

$$M(n) = \max\{\nu(C) : C \text{ is a cylinder of length } n,\ \nu(C) > 0\}$$

leads to a phase transition in the family of generalized entropies at $q = 1$. As the next example shows, the sup-exponential behavior of

$$m(n) = \min\{\nu(C) : C \text{ is a cylinder of length } n,\ \nu(C) > 0\}$$

leads to a phase transition at $q = 0$.

*Example 2.29.* This example is taken from [1]. Consider a piecewise monotonic transformation of a unit interval

$$f(x) = \sqrt{1 - |1 - 2x|}.$$

This transformation has an absolutely continuous invariant measure $\mu$:

$$\mu([0, x]) = x^2.$$

A natural partition in this case is $\xi = \{I_0, I_1\} = \{[0, 1/2), [1/2, 1]\}$. As in the previous example consider the symbolic space $X = \{0, 1\}^{\mathbb{Z}_+}$, and the shift invariant measure $\nu$, which is obtained from $\mu$ as in (2.21). Contrary to the previous example, one can show that the measure of the cylinder $C_0^n(0, 0, \dots, 0)$ decays faster than any exponential function of $n$

$$\liminf_{n\to\infty} -\frac{1}{n} \log \nu(C_0^n(0, 0, \dots, 0)) = +\infty.$$

Hence, using the one-sided analogue of the first inequality in (2.11), for $q < 0$ one gets

$$\begin{aligned} I(q, 2^{-(k+1)}, n) &\geq \frac{1}{(1 + |q|)n} \log \frac{1}{\nu(C_0^{n+k-1}(0, \dots, 0))^{|q|}} \\ &= -\frac{|q|}{(1 + |q|)n} \log \nu(C_0^{n+k-1}(0, \dots, 0)), \end{aligned} \tag{2.22}$$

and hence

$$\underline{H}_\nu(\sigma, q) = +\infty.$$

From this we conclude that

$$q_* = \inf\{q \in \mathbb{R} : \overline{H}_\nu(\sigma, q) < \infty\} = 0.$$

The fact that in both examples we switched to a symbolic representation, is not the source of the pathology. With a bit more refined analysis we could establish the same results for $H_\mu(f, q)$ directly.

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# Chapter 3

# Rényi entropies

We study the generalized Rényi entropies which were introduced in the physics literature. The proper definition of these entropies needs, in our opinion, further clarification. We show that the original Rényi entropies do not contain any new information. On the other hand, we introduce the notion of the correlation entropies, which are not invariants of the dynamical systems in the measure–theoretic sense, but they are invariant under transformations of the state space with bounded distortion. With these correlation entropies we provide a formal definition which can serve as a basis for the results reported in the physics literature.

## 3.1 Introduction

The notion of a generalized spectrum for dimensions was introduced in [15] for the characterization of fractals and strange attractors. Two approaches were considered. One named after Rényi, another based on correlation integrals and sometimes named after Hentschel–Procaccia. Let $\mu$ be some invariant measure on an attractor in $\mathbb{R}^m$ (in practice, one thinks of an invariant ergodic Sinai-Ruelle-Bowen measure).

**Definition 3.1.** *For given $q \geq 0$, $q \neq 1$, define the following spectra (assuming all the limits exist)*

$$\text{\textit{Rényi}}: \qquad D_q = \frac{1}{q-1} \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \sum_i p_i^q,$$

*where $p_i = \mu($ $i$-th cell of a partition by cubes of size $\varepsilon$ $)$,*

$$\text{\textit{Hentschel-Procaccia}}: \qquad \tilde{D}_q = \frac{1}{q-1} \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \int \mu(\mathcal{B}(x,\varepsilon))^{q-1}\, d\mu(x),$$

*where $\mathcal{B}(x,\varepsilon)$ is an open $\varepsilon$-ball around $x$.*

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This chapter is based on F. Takens, E. Verbitski *Generalized entropies: Rényi and correlation integral approach*, Nonlinearity **11**, 1998, pp.771-782.

In these formulae the expression, of which the limit is taken, can be interpreted as $L_{q-1}(\mathbb{R}^m, \mu)$ averages of the following measurable functions:

$$F_R(x) := \mu(C(x)),$$

where $C(x)$ is the cube of size $\varepsilon$ containing $x$ in the case of the Rényi spectrum, and

$$F_{HP}(x) := \mu(\mathcal{B}(x, \varepsilon))$$

in the case of Hentschel-Procaccia spectrum. The value $F_R(x)$ is the probability for a $\mu$-random point to fall into the same cell with $x$, $F_{HP}(x)$ is the probability of an $\varepsilon$-neighborhood of x. These generalized dimensions were actively studied in the physics literature; for a good mathematical description of various definitions we refer to [19]. In particular, the two spectra $D_q$ and $\tilde{D}_q$ are shown to be equal for every $q > 1$.

Working with box-partitions may seem arbitrary. A natural question is for example, whether the definition of Rényi dimensions is invariant under translations of the grids? It was pointed out in [25, 14] that the generalized dimensions are invariant under smooth transformation of the state space.

The dynamical counterparts of generalized dimensions – generalized entropies – appeared in [13], where only the first (Rényi) approach was considered. Since then the Rényi entropies of dynamical systems were studied by various authors in the physics literature. Introduced for the purposes of the numerical characterization of sensitive dependence on initial condition of the chaotic data [9, 12, 14], the notion of the Rényi entropies got further development in works [1, 3, 8], where the relations to the thermodynamic and multifractal formalisms of dynamical systems have been established. Rényi entropies have been applied to a characterization of intermittency in [2, 18]. Phase transitions in the spectrum of Rényi entropies have been found for some maps [7, 6, 16, 24].

Some authors claim that the Rényi entropies at $q = 0$ and $q = 1$ are the topological and measure–theoretic entropies respectively. This however leads to a paradox, since by preserving the measure–theoretical structure of a dynamical systems we can change the topological structure in such way that the topological entropy of a new dynamical system will be different.

In later publications [1, 9] more general definitions of the Rényi entropies were introduced. In the present paper we adopt the definition from [1]. This definition is completely analogous to the standard definition the measure–theoretic entropy, except that the Shannon information function, defined on discrete probability distributions $p = (p_1, p_2, \dots)$ with $p_i \geq 0$ and $\sum_i p_i = 1$,

$$H(p) = -\sum_i p_i \log p_i$$

is substituted by a more general Rényi information function of order $q$, $q \neq 1$

$$H_q(p) = -\frac{1}{q-1} \log\bigl(\sum_i p_i^q\bigr).$$

The Shannon information function can and will be included in this family for $q = 1$ so as to make this family continuous. The precise definition is given in the next section.

The Rényi entropies, defined in this way, are measure–theoretic invariants, i.e., two dynamical systems, isomorphic in the measure–theoretical sense, have equal Rényi entropies. The definition of the Rényi entropies from [9] uses only a measure–theoretical structure as well, but the construction requires the existence of a certain limit when the diameter of the partition tends to zero. It follows from our result that such limit does not exist in general.

The main reason why one might argue if the definition from [1] is the right one is:

**Theorem 3.2.** *For an ergodic dynamical system $(X, \mathfrak{B}, \mu, T)$ with entropy $h(T) > 0$, the Rényi entropies (defined in section 3.2) are given by the following formula*

$$h(T, q) = \begin{cases} +\infty, & q < 1 \\ h(T), & q \geq 1 \end{cases} \tag{3.1}$$

This type of result was to be expected, at least for Bernoulli shifts, because up to measure–theoretic isomorphisms the entropy is the complete invariant for them. On the contrary, working with a smaller class of partitions, like a class of partitions onto cylinders in the case of symbolic dynamical systems, one obtains the Rényi entropies which cannot be derived from the entropy. This shows that the partitions leading up to the supremum are rather pathological in the geometric sense.

In order to extract new information about the dynamics from the generalized entropies we cannot ignore the geometry (topology). The family of the correlation entropies, defined below, is suitable for these purposes.

The definition of these correlation entropies is inspired by [9, 14, 11, 25]. The correlation entropies, however, are not measure–theoretical invariants, due to the fact that their definition uses the notion of distance. They are invariant under measure-preserving bijections of bounded distortion, i.e., under isomorphisms $f : (X_1, d_1) \to (X_2, d_2)$ such that for some $K > 1$

$$\frac{1}{K} < \frac{d_2(f(x), f(y))}{d_1(x, y)} < K \quad \forall x, y \in X_1.$$

**Definition 3.3.** *Let $(X, d)$ be a compact metric space, $T : X \to X$ is a continuous map, preserving a Borel non-atomic probability measure $\mu$. For each $q \in \mathbb{R}$ we define the correlation entropy of order $q$ as follows*

$$C(T, q) = \lim_{\varepsilon \to 0} \liminf_{k \to \infty} \frac{1}{k} I(q, k, \varepsilon),$$

*with*

$$I(q, k, \varepsilon) = \begin{cases} -\frac{1}{q-1} \log \int \mu(\mathcal{B}_k(x, \varepsilon))^{q-1} \, d\mu, & q \neq 1, \\ -\int \log \mu(\mathcal{B}_k(x, \varepsilon)) \, d\mu, & q = 1, \end{cases}$$

*where $\mathcal{B}_k(x, \varepsilon) = \{y \in X : \ d(T^i x, T^i y) < \varepsilon \text{ for } i = 0, \ldots, k-1\}$.*

The above definitions for $q = 1$ and $q \neq 1$ are inspired by the Shannon and Rényi information functions, respectively. Using the Brin–Katok formula [4] and Fatou's theorem it is easy to see that $C(T, 1)$ is equal to the measure–theoretic entropy. For completeness we give the formulae for the correlation entropies in the case of symbolic dynamical systems and some specific measures. Let $X = \Omega^{\mathbb{Z}}$, where $\Omega$ is a finite alphabet, equiped with a standard metric $d(x, y) = 2^{-N}$ where

$$N = \min\{|i| : \ x_i \neq y_i\}$$

and the dynamics is given by the left shift $T$, i.e. $T(x) = y$ with $y_i = x_{i+1}$. Then (see chapter 2)

(1) for the Bernoulli measure $\mu$ generated by probabilities $(p_1, \ldots, p_m)$ the correlation entropies are given by the following formula

$$C(T, q) = -\frac{1}{q-1} \log\left(p_1^q + \ldots + p_m^q\right), \quad q \neq 1;$$

(2) for the Gibbs measure $\mu$ with a potential $\varphi$ the correlation entropies can be written in terms of the topological pressure $P$ [1],

$$C(T, q) = -\frac{P(q\varphi) - qP(\varphi)}{q-1}, \quad q \neq 1.$$

The particularity of the symbolic dynamical systems is the following equality

$$\mathcal{B}_n(x, 1/2^k) = C_{-k}^{n+k-1}(x) \quad \text{ for all } x \in X \text{ and } k, n \in \mathbb{N},$$

where $C_{-k}^{n+k-1}(x)$ is the cylinder determined by coordinates $x_{-k}, \ldots, x_{n-k+1}$. This equality implies that for every $q$ the correlation entropy and the Rényi entropy with respect to the partition into cylinders coincide.

Correlation entropies given by (1) and (2) are real–analytic with respect to $q$ [21] and $C(T, 0) = h_{top}(T)$, where $h_{top}(T)$ is the topological entropy of $T$.

Correlation entropies are closely related to a reconstruction technique of nonlinear time series analysis and their definition can be directly implemented as numerical procedures. Another advantage of this definition is that partitions are excluded from the consideration, which in applications allows to save essential memory resources compared to the usual box-counting algorithms

## 3.2 Definition of the Rényi entropies

We shall consider Lebesgue probability spaces, for the details see [5, appendix 1]. We consider invertible measure–preserving dynamical systems.

**Definition 3.4.** *A map $T$ defined on a Lebesgue probability space $(X, \mathfrak{B}, \mu)$ is called an automorphism if*

*(1) $T$ is a one-to-one map of $X$ onto itself,*

*(2) for all $A \in \mathfrak{B}$ we have $TA, T^{-1}A \in \mathfrak{B}$ and $\mu(A) = \mu(TA) = \mu(T^{-1}A)$.*

Suppose $\xi$ is a partition of the space $X$ into measurable sets $\{\Delta_1, \dots, \Delta_n\}$, i.e.,

(1) $\cup_{i=1}^n \Delta_i = X$,

(2) $\Delta_i \cap \Delta_j = \varnothing$ for $i \neq j$.

For any $q \in \mathbb{R}$ the entropy of order $q$ of the partition $\xi$ is the number

$$H_q(\xi) = \begin{cases} -\dfrac{1}{q-1} \log\Bigl(\sum\limits_{i=1}^n \mu(\Delta_i)^q\Bigr), & \text{for } q \neq 1, \\ -\sum\limits_{i=1}^n \mu(\Delta_i) \log \mu(\Delta_i), & \text{for } q = 1, \end{cases} \tag{3.2}$$

with a standard agreement $0^\alpha = 0$ for all $\alpha \in \mathbb{R}$ and $0 \log 0 = 0$.

It is easy to check the following monotonicity property

$$H_{q_1}(\xi) \leq H_{q_2}(\xi) \text{ for any } \xi \text{ and } q_1 \geq q_2.$$

**Definition 3.5.** *The Rényi entropy of order $q$ with respect to a partition $\xi$ is*

$$h(T, q, \xi) = \liminf_{k \to \infty} \frac{1}{k} H_q\left(\xi^{(k)}\right), \tag{3.3}$$

*where $\xi^{(k)} = \xi \vee T^{-1}\xi \vee \ldots \vee T^{-k+1}\xi$ is the partition on sets $\bigcap_{i=0}^{k-1} T^{-i}\Delta_{j_i}$ with $\Delta_{j_i} \in \xi$.*

*Remark.* For $q = 1$ it is proven (see for example [5, 23]) that the limit in (3.3) exists. The proof of this result is based on a so called subadditivity property of the entropy function $H_1$, i.e.,

$$H_1(\xi \vee \eta) \leq H_1(\xi) + H_1(\eta)$$

for all partitions $\xi, \eta$. It was shown by Rényi [20] that for $q \neq 1$ the entropy function $H_q$ has no longer the subadditivity property. Therefore, the existence of the limit for $q \neq 1$ in general is not proved.

**Definition 3.6.** *The Rényi entropy of order $q$ of an automorphism $T$ is the number*

$$h(T, q) = \sup_\xi h(T, q, \xi),$$

*where the supremum is taken over all finite partitions $\xi$ of $X$.*

**Proposition 3.7.** *One has the following properties of the Rényi entropies:*

*1) $h(T, q) \geq 0$ for all $q$;*

*2) $h(T, q_1) \geq h(T, q_2)$ for $q_1 \leq q_2$;*

*3) $h(T, 1) = h(T)$, where $h(T)$ is the measure–theoretic (or Kolmogorov–Sinai) entropy.*

## 3.3 Rényi entropies as metric invariants

In this section we show that the Rényi entropies are invariant under (weak) isomorphisms of dynamical systems.

**Definition 3.8.** *Let $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ be Lebesgue probability spaces, a map $\varphi$ is said to be a homomorphism if it is defined mod $0$, i.e., there exist sets $X_1 \subseteq X$, $Y_1 \subseteq Y$ of full measure ($\mu(X_1) = 1$, $\nu(Y_1) = 1$), such that $\varphi$ maps $X_1$ onto $Y_1$, and*

*1) $\varphi$ is measurable, i.e., for any $C \in \mathfrak{D}$ we have $\varphi^{-1}(C) \in \mathfrak{B}$*

*2) $\varphi$ preserves measure, i.e., for any $C \in \mathfrak{D}$ $\nu(C) = \mu(\varphi^{-1}(C))$.*

Let $T_i$ be an automorphism of a Lebesgue space $(X_i, \mathfrak{B}_i, \mu_i)$, $i = 1, 2$. We say that $T_2$ is a factor (or, is a homomorphic image) of $T_1$ if there exists a homomorphism $\varphi : X_1 \to X_2$ such that

$$\varphi T_1 = T_2 \varphi \quad \mu_1\text{-a.s. on } X_1.$$

**Definition 3.9.** *Two automorphisms $T_1$ and $T_2$ are called weakly isomorphic if $T_2$ is a factor $T_1$ and $T_1$ is a factor of $T_2$.*

*$T_1$ is metrically isomorphic (or just isomorphic) to $T_2$ if there exists an isomorphism $\varphi$ (i.e., $\varphi$ is one-to-one homomorphism, and its inverse $\varphi^{-1}$ is also a homomorphism) such that*

$$\varphi T_1 = T_2 \varphi \quad \mu_1\textit{-a.s. on } X_1.$$

Suppose that $T_2$ is a factor of $T_1$ under a homomorphism $\varphi$. Then for an arbitrary finite partition $\xi$ of $X_2$ we have

$$H_q\Big(\bigvee_{i=0}^{k-1} T_2^{-i}\xi\Big) = H_q\Big(\bigvee_{i=0}^{k-1} \varphi^{-1}T_2^{-i}\xi\Big) = H_q\Big(\bigvee_{i=0}^{k-1} T_1^{-i}\varphi^{-1}\xi\Big).$$

Hence, $h(T_2, q, \xi) = h(T_1, q, \varphi^{-1}\xi)$. Therefore

$$h(T_2, q) = \sup_{\xi} h(T_2, q, \xi) = \sup_{\xi} h(T_1, q, \varphi^{-1}\xi) \leq h(T_1, q).$$

This implies:

**Proposition 3.10.** *If $T_2$ is a factor of $T_1$ then for any $q$*

$$h(T_2, q) \leq h(T_1, q).$$

As a corollary we have:

**Proposition 3.11.** *If $T_1$ and $T_2$ are (weakly) isomorphic then for any $q$*

$$h(T_2, q) = h(T_1, q).$$

## 3.4 Rényi entropies of Bernoulli shifts with respect to partitions into cylinders

Let $\Omega = \{1, \dots , m\}$ be a finite alphabet. Let $X = \{\mathbf{x} = \{x_i\}_{i=-\infty}^{\infty} : \ x_i \in \Omega\}$ and $T$ be a left shift:

$$T(\mathbf{x})_i = x_{i+1}.$$

For any $s \le t$, and sequence $\{a_s, \dots , a_t\}$, with $a_i \in \Omega$, we define a cylinder

$$C_s^t(a_s, \dots , a_t) = (\mathbf{x} \in X : \ x_i = a_i \text{ for } i = s, \dots , t).$$

We consider the Borel $\sigma$-algebra $\mathfrak{B}(X)$ with respect to the metric, which is given by $d(x, y) = 2^{-N}$, where $N = \min\{|i| : x_i \neq y_i\}$. One can show that $\mathfrak{B}(X)$ is the minimal $\sigma$-algebra containing all cylinders.

Let $p = (p_1, \dots , p_m)$ be a probability vector, i.e., $p_i \ge 0$ for any $i$ and $\sum p_i = 1$. We define the measure $\rho = \rho(p)$ on $\Omega$ by setting $\rho(\{i\}) = p_i$. Let $\mu = \mu(p)$ be the corresponding product measure on $X = \Omega^{\mathbb{Z}}$.

**Definition 3.12.** *The transformation $T$ of $(X, \mathfrak{B}(X), \mu)$, as defined above, is called a Bernoulli shift.*

We now come to the computation of the Rényi entropy of the Bernoulli shift $T$ with respect to partitions into cylinders.

For any $s, t$, $s \le t$, and any set $\{a_s, \dots , a_t\}$, the measure of the corresponding cylinder is given by

$$\mu\left(C_s^t(a_s, \dots , a_t)\right) = \prod_{i=s}^{t} p_{a_i}.$$

Consider an arbitrary partition $\xi$ into cylinders

$$\xi = \{C_s^t(a_s, \dots , a_t) : \ a_i \in \Omega, \ s \le i \le t\},$$

for some $s \le t$. It is easy to see that

$$H_q(\xi^{(k)}) = -(t - s + k)\frac{1}{q-1}\log\Big(p_1^q + \dots + p_m^q\Big) \text{ for } q \neq 1.$$

Therefore, for any partition into cylinders $\xi$ one has

$$h(T, q, \xi) = -\frac{1}{q-1}\log\Big(p_1^q + \dots + p_m^q\Big), \qquad \text{for } q \neq 1, \tag{3.4}$$

$$h(T, 1, \xi) = -p_1 \log p_1 - \dots - p_m \log p_m. \tag{3.5}$$

Since any partition into cylinders is generating, the measure–theoretic entropy is equal to the entropy with respect to this partition. Therefore, for $q = 1$, one has

$$h(T) = h(T, 1) = h(T, 1, \xi) = -p_1 \log p_1 - \dots - p_m \log p_m.$$

## 3.5 Rényi entropies of Bernoulli automorphisms

**Definition 3.13.** *An automorphism $T$ on $(X, \mathfrak{B}, \mu)$ is called a Bernoulli automorphism if it is isomorphic to some Bernoulli shift.*

Suppose we have two Bernoulli automorphisms $T_1$ and $T_2$ with equal measure–theoretic entropies. Then

(1) by Sinai's theorem [22] they are weakly isomorphic;

(2) by Ornstein's theorem [17] they are metrically isomorphic.

Since the Rényi entropy of any order $q$ is invariant under (weak) isomorphisms, for Bernoulli automorphisms $T_1$ and $T_2$ we have

$$h(T_1, 1) = h(T_2, 1) \;\Rightarrow\; h(T_1, q) = h(T_2, q) \quad \text{for all } q.$$

Another theorem, which is important for us, is due to Sinai [22]:

**Theorem 3.14.** *Let $T$ be an arbitrary ergodic automorphism of some Lebesgue space $(X, \mathfrak{B}, \mu)$. Each Bernoulli automorphism $T_1$ with $h(T_1) \le h(T)$ is a factor of the automorphism $T$.*

*Remark.* In other words, this theorem says that there exists a partition of the phase space such that the symbolic dynamics defined with respect to this partition is a Bernoulli shift. Such partitions can be pathological from the geometric point of view. On the other hand, it is known that even for hyperbolic dynamical systems such 'good' objects as Markov partitions can have a complicated geometrical structure, for example, the boundaries of such Markov partitions may not be piecewise smooth.

The following statement is a simple corollary of the formula (3.4).

**Proposition 3.15.** *Let $T$ be an arbitrary ergodic automorphism with $h(T) \ge \log M$, where $M$ is an integer, $M > 1$. Then $h(T, q) \ge \log M$ for all $q \ge 1$.*

*Proof.* Consider a shift $\tilde{T}$ over all infinite sequences from the alphabet $\Omega = \{1, \ldots, M\}$ with the corresponding Bernoulli measure generated by $p_1 = \ldots = p_M = 1/M$. It is easy to see that $h(\tilde{T}) = \log M$. Applying Theorem 3.14 we conclude that $\tilde{T}$ is a factor of $T$. Making use of Proposition 3.10 we get that for all $q \ge 1$

$$h(T, q) \ge h(\tilde{T}, q).$$

Then applying formula (3.4) and using the monotonicity properties from Proposition 3.7 we conclude that for $q \ge 1$

$$\log M = h(\tilde{T}, 1) \ge h(\tilde{T}, q) \ge h(\tilde{T}, q, \xi) = \log M,$$

where $\xi$ is an arbitrary partition into cylinders. Hence $h(T, q) \ge \log M$ for all $q \ge 1$. $\square$

## 3.6 Rényi entropies for $q < 1$

In this section we prove the following theorem

**Theorem 3.16.** *Let $T$ be an ergodic automorphism of a Lebesgue space $(X, \mathfrak{B}, \mu)$ with $h(T) > 0$. Then for $q < 1$ we have*

$$h(T, q) = +\infty.$$

We need a few preliminary lemmas:

**Lemma 3.17.** *For any $h$, $0 < h < \infty$, there exist $m \in \mathbb{N}$ and $p = (p_1, \dots, p_m)$, $p_i > 0$, such that*

*1)* $\sum p_i = 1$,

*2)* $-\sum p_i \log p_i = h$.

*Proof.* The function $-\sum p_i \log p_i$ maps an open $m$-dimensional simplex $S_m^o = \{p = (p_1, \dots, p_m) : p_i > 0, \sum p_i = 1\}$ onto $(0, \log m]$. Hence for sufficiently large $m$ $(\log m > h)$, there exists $p \in S_m^o$ with entropy $h$. □

**Lemma 3.18.** *For any $h$, $0 < h < \infty$, and $s \in (0, 1]$, with $h > -s \log s$, there exist $m \in \mathbb{N}$ and $p = (p_1, \dots, p_m)$, $p_i > 0$, such that*

*1)* $\sum p_i = s$,

*2)* $-\sum p_i \log p_i = h$.

*Proof.* Let $\hat{h} = \dfrac{h}{s} + \log s > 0$. Then by Lemma 3.17 there exist $m$ and $\hat{p} = (\hat{p}_1, \dots, \hat{p}_n)$ such that $\sum \hat{p}_i = 1$ and $-\sum \hat{p}_i \log \hat{p}_i = \hat{h}$. Let $p_i = s\hat{p}_i$, for $i = 1, \dots, m$. Then $\sum p_i = s$ and

$$-\sum p_i \log p_i = -s \sum \hat{p}_i \log \hat{p}_i - s \log s = s\hat{h} - s \log s = h.$$

□

<u>*Proof of Theorem 3.16.*</u> If $h(T) = +\infty$ then there is nothing to prove, since for $q < 1$ $h(T, q) \geq h(T, 1) = h(T)$.

Consider now the case $0 < h(T) < \infty$. Denote for simplicity $h(T)$ by $h$. Let

$$\mathfrak{M}(h) = \Big\{ p = (p_1, \dots, p_m) : \; p_i > 0, \; \sum p_i = 1, \; -\sum p_i \log p_i = h \Big\}.$$

For each $p \in \mathfrak{M}(h)$ we can construct a Bernoulli shift with the given probabilities $\{p_i\}$. All these shifts have equal measure-theoretic entropies, therefore they are (weakly) isomorphic and have equal Rényi entropies for all $q$.

We construct an element of $\mathfrak{M}(h)$ with some specific properties. For $q \in (0,1)$ and $n \in \mathbb{N}$ we define $p_1 = \ldots = p_n = n^{-\frac{1+q}{2q}}$. Then

$$p_1 + \ldots + p_n = n^{-\frac{1-q}{2q}} \to 0 \text{ as } n \to \infty,$$
$$-p_1 \log p_1 - \ldots - p_n \log p_n = \frac{1+q}{2q} n^{-\frac{1-q}{2q}} \log n \to 0 \text{ as } n \to \infty.$$

Let $s = 1 - n^{-\frac{1-q}{2q}}$, $\tilde{h} = h - \frac{1+q}{2q} n^{-\frac{1-q}{2q}} \log n$. Then for sufficiently large $n$ we have

$$s > 0, \quad \tilde{h} > 0, \quad \text{and} \quad \tilde{h} > -s \log s > 0.$$

By Lemma 3.17 there exist $k \in \mathbb{N}$ and $v = (v_1, \ldots, v_k)$ with $v_j > 0$ such that

$$\sum_j v_j = s, \quad -\sum_j v_j \log v_j = \tilde{h}.$$

Hence, for the positive vector $p = (p_1, \ldots, p_n, v_1, \ldots, v_k) =: (p_1, \ldots, p_{n+k})$ we have

$$\sum_i p_i = 1, \quad -\sum_i p_i \log p_i = h.$$

Thus $p \in \mathfrak{M}(h)$. Let $T'$ be a Bernoulli shift corresponding to the probability vector $p$. Since $h(T') = h(T)$, we can conclude by the Sinai's theorem (Theorem 3.14) that $T'$ is a factor of $T$. Since the Rényi entropy does not increase when taking factors (Proposition 3.10), we have

$$h(T,q) \geq h(T',q) \geq h(T',q,\xi) = \frac{1}{1-q} \log\Big(p_1^q + \ldots + p_n^q + p_{n+1}^q + \ldots + p_{n+m}^q\Big)$$
$$\geq \frac{1}{1-q} \log\Big(p_1^q + \ldots + p_n^q\Big) = \frac{1}{1-q} \log\Big(n^{\frac{1-q}{2}}\Big) = \frac{1}{2} \log n,$$

where $\xi$ is a partition by cylinders as in section 3.4.

Since $n$ is an arbitrary integer we have $h(T,q) = \infty$ for any $0 < q < 1$. Since for all $q \leq 0$ one has $h(T,q) \geq h(T,0.5) = +\infty$, we proved the statement. $\square$

## 3.7 Rényi entropies for $q \geq 1$

The main result of this section is

**Theorem 3.19.** *Let $T$ be an ergodic automorphism of a Lebesgue space $(X, \mathfrak{B}, \mu)$. Then for $q \geq 1$*

$$h(T,q) = h(T).$$

We need a few preliminary statements.

**Lemma 3.20.** *Let $T$ be a measure-preserving automorphism of $(X, \mathfrak{B}, \mu)$. Let $\xi$ and $\eta$ be finite partitions of $(X, \mathcal{B})$. If $\xi$ is a refinement of $\eta$, i.e., each element of $\eta$ is a union of elements of $\xi$, then for every $q \in \mathbb{R}$*

$$H_q(\eta) \leq H_q(\xi),$$

*and hence*

$$h(T, q, \eta) \leq h(T, q, \xi).$$

*Proof.* Obvious. □

**Lemma 3.21.** *Let $T$ be a automorphism of $(X, \mathfrak{B}, \mu)$. Then for any $q \in \mathbb{R}$ and $m \in \mathbb{N}$ one has*

$$h(T^m, q) = mh(T, q).$$

*Proof.* We first show the inequality $h(T^m, q) \geq mh(T, q)$. For this consider an arbitrary finite partition $\xi$. Then

$$\begin{aligned} h\Big(T^m, q, \bigvee_{i=0}^{m-1} T^{-i}\xi\Big) &= \liminf_{k\to\infty} \frac{1}{k} H_q\left(\bigvee_{j=0}^{k-1} T^{-mj}\left(\bigvee_{i=0}^{m-1} T^{-i}\xi\right)\right) \\ &= m \liminf_{k\to\infty} \frac{1}{km} H_q\left(\bigvee_{i=0}^{km-1} T^{-i}\xi\right) \\ &\geq m \liminf_{n\to\infty} \frac{1}{n} H_q\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right) = mh(T, q, \xi) \end{aligned}$$

Taking the supremum over all finite partitions we obtain the inequality

$$h(T^m, q) \geq mh(T, q).$$

So now we have to prove the opposite inequality. By definition

$$\begin{aligned} h(T^m, q, \xi) &= \liminf_{k\to\infty} \frac{1}{k} H_q\Big(\xi \vee T^{-m}\xi \vee \ldots \vee T^{-m(k-1)}\xi\Big) \\ &= \lim_{n\to\infty} \inf_{k\geq n} \frac{1}{k} H_q\Big(\xi \vee T^{-m}\xi \vee \ldots \vee T^{-m(k-1)}\xi\Big) \end{aligned}$$

Consider any $k \in \mathbb{N}$. Then it is clear that

$$\bigvee_{i=0}^{n-1} T^{-i}\xi \ \text{ is a refinement of } \ \xi \vee T^{-m}\xi \vee \ldots \vee T^{-m(k-1)}\xi \tag{3.6}$$

for $n = km, \ldots, km + m - 1$. By Lemma 3.20 this implies an inequality for entropies of order $q$:

$$\begin{aligned} \frac{1}{k} H_q\Big(\xi \vee T^{-m}\xi \vee \ldots \vee T^{-m(k-1)}\xi\Big) &\leq \frac{1}{k} H_q\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) = \frac{n}{k}\frac{1}{n} H_q\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) \\ &\leq \frac{km+m-1}{k}\,\frac{1}{n} H_q\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) \leq m\Big(1 + \frac{1}{k}\Big)\frac{1}{n} H_q\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) \end{aligned} \tag{3.7}$$

for $n = km, \dots, km+m-1$. We introduce the following notation:

$$c_k = \frac{1}{k} H_q\Big(\xi \vee T^{-m}\xi \vee \ldots \vee T^{-m(k-1)}\xi\Big), \quad \text{and} \quad a_n = \frac{1}{n} H_q\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big).$$

Then (3.7) can be rewritten in the following form

$$c_k \leq m\Big(1+\frac{1}{k}\Big)a_n \tag{3.8}$$

for $n = km, \dots, km+m-1$. Taking the infimum in (3.8) we obtain

$$\inf_{l\geq k} c_l \leq m \inf_{l \geq k}\Big(1+\frac{1}{l}\Big) \inf_{n=lm,\dots,lm+m-1} a_n \leq m\Big(1+\frac{1}{k}\Big) \inf_{n\geq km} a_n.$$

From this one can conclude that

$$\liminf_{k\to\infty} c_k \leq m \liminf_{n\to\infty} a_n,$$

which is equivalent to

$$h(T^m, q, \xi) \leq m h(T, q, \xi).$$

This finishes the proof of lemma. $\square$

The following is a weaker version of Theorem 3.19:

**Theorem 3.22.** *Let $T$ be an ergodic automorphism of $(X, \mathfrak{B}, \mu)$, such that $T^m$ is ergodic for every $m \in \mathbb{N}$. Then*

$$h(T,q) = h(T) \textit{ for all } q \geq 1.$$

*Proof.* Assume that there exists $q > 1$ such that

$$h(T) - h(T,q) > 0.$$

Let us consider $T^m$. Then

$$h(T^m) - h(T^m, q) = m(h(T) - h(T,q)) \to \infty \quad \text{ as } m \to \infty.$$

Therefore, for sufficiently large $m$ there exists $M \in \mathbb{N}$ such that

$$mh(T,q) = h(T^m, q) < \log M < h(T^m) = mh(T). \tag{3.9}$$

Applying Proposition 3.15 to an ergodic automorphism $T^m$ we conclude that for all $q \geq 1$

$$h(T^m, q) \geq \log M. \tag{3.10}$$

So, comparing (3.9) and (3.10) we arrive at a contradiction, and the Theorem is proved. $\square$

*Proof of Theorem 3.19.* If $h(T) = 0$ then there is nothing to prove since

$$0 = h(T) = h(T, 1) \geq h(T, q) \geq 0.$$

Suppose now $0 < h(T) < \infty$. Since $T$ is an ergodic automorphism, it has a factor which is a Bernoulli automorphism $T'$ with entropy $h(T') = h(T)$. Bernoulli automorphisms are mixing. Hence, $T'^m$ is an ergodic automorphism for each $m$ [10, Prop.4.7]. This means that we can apply Theorem 3.22 to $T'$. Therefore for $q \geq 1$

$$h(T', q) = h(T', 1) = h(T') = h(T).$$

Since $T'$ is a factor of $T$ we have for $q \geq 1$

$$h(T) = h(T, 1) \geq h(T, q) \geq h(T', q) = h(T') = h(T).$$

If the automorphism $T$ has infinite entropy, then by Proposition 3.15 $h(T, q) \geq \log M$ for every integer $M$. Therefore, $h(T, q) = +\infty$, i.e., $h(T, q) = h(T)$ for all $q \geq 1$. This finishes the proof. □

## 3.8 Remarks

(1) Theorem 3.2 is valid for non-invertible measure–preserving maps $T$ as well. A key instrument of our proof – Theorem 3.14 – has its analogue [22] for ergodic endomorphisms of Lebesgue spaces. The proof of our theorem remains the same.

(2) The condition of positive entropy in Theorem 3.2 can be weakened as well. The result remains true for aperiodic measure–preserving automorphisms. The proof is based on a completely different technique (Rokhlin–Halmos towers) and will appear elsewhere.

(3) The elements of partitions used in the proof can be uneven in size, both in measure and geometry. One should not think that this is the source of the pathology in the spectrum of Rényi entropies. In order to approach the supremum one does not have to consider such uneven partitions. Indeed, if $\xi$ is an arbitrary partition, then by subdividing the elements of $\xi$ one can obtain a new partition $\tilde{\xi}$, the elements of which have approximately equal measures. Since $\tilde{\xi}$ is a refinement of $\xi$, the Rényi entropy with respect to $\tilde{\xi}$ can only be larger (Lemma 3.20). If the dynamical system acts on a metric space $(X, d)$, then the same sort of arguments allow us to construct partitions with uniformly bounded diameters as well.

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# Chapter 4

# Rényi entropies of aperiodic dynamical systems

In this chapter we continue the study of Rényi entropies of measure-preserving transformations started in the previous chapter. We have established there that for ergodic transformations with positive entropy, the Rényi entropies of order $q$, $q \in \mathbb{R}$, are equal to either plus infinity ($q < 1$), or, to the measure-theoretic (Kolmogorov-Sinai) entropy ($q \geq 1$). The answer for non-ergodic transformations is different: the Rényi entropies of order $q > 1$ are equal to the essential infimum of the measure-theoretic entropies of measures forming the decomposition into ergodic components. Thus, it is possible that the Rényi entropies of order $q > 1$ are strictly smaller than the measure-theoretic entropy, which is the average value of entropies of ergodic components.

This result is a bit surprising: the Rényi entropies are metric invariants, which are sensitive to ergodicity.

The proof of the described result is based on the construction of partitions with independent iterates. However, these partitions are obtained in different ways depending on $q$: for $q > 1$ we prove a version of the well-known Sinai theorem on Bernoulli factors for non-ergodic transformations; for $q < 1$ we use the notion of collections of independent sets in Rokhlin-Halmos towers and their properties.

## 4.1 Introduction

Alfred Rényi introduced the generalization of the Shannon information (entropy) in the beginning of sixties. His approach was based on an axiomatic definition of information, and consisted of including the standard entropy function

$$H(p_1, \dots, p_n) = -\sum_{i=1}^{n} p_i \log p_i,$$

The chapter is based on F. Takens, E. Verbitski, *Rényi entropies of aperiodic dynamical systems*, preprint, 1999.

into a one-parameter family of generalized entropy functions

$$H_q(p_1,\dots,p_n) = -\frac{1}{q-1}\log\Bigl(\sum_{i=1}^{n} p_i^q\Bigr), \quad q\neq 1.$$

For a fixed probability distribution $(p_1,\dots,p_n)$ the standard entropy is recovered from the generalized entropies as follows

$$H(p_1,\dots,p_n) = \lim_{q\to 1} H_q(p_1,\dots,p_n)$$

Since then the Rényi entropies have been successfully used in information theory and statistics, and more recently in thermodynamics and quantum mechanics. In dynamical systems, Hentschel and Procaccia [8] suggested a one-parameter family of generalized dimensions based on Rényi's approach. These dimensions proved to be extremely useful in problems of multifractal analysis and characterization of chaotic attractors, see e.g. [13].

Some attempts [7], [6] were made to introduce the generalized entropies of dynamical system using Rényi's approach. The idea was to produce a sufficiently rich family of invariants of a dynamical system, which will take into account the non-uniform behavior of invariant measures. However, the proposed way of generalizing the Kolmogorov-Sinai entropy using $H_q$ instead of $H_1$, turned out to be non-productive. In [22] we have established the following fact.

**Theorem 4.1.** *For an ergodic dynamical system $(X,\mathfrak{B},\mu,T)$ with positive measure–theoretic entropy $h(T,\mu)>0$, the Rényi entropies are given by the following formula*

$$h(T,q,\mu) = \begin{cases} +\infty, & q<1,\\ h(T,\mu), & q\geq 1.\end{cases}$$

Also in [22] we suggested another family of generalized entropies, which recovers the results reported in the physics literature [1].

The proof of Theorem 4.1 relies heavily on Sinai's theorem on Bernoulli factors [19], for which the assumptions of ergodicity and positiveness of the measure–theoretic entropy are crucial.

In this paper we prove a result, similar to Theorem 4.1, but without the above assumptions. We consider aperiodic measure-preserving automorphisms, i.e., transformations $T$ of some Lebesgue space $(X,\mathfrak{B},\mu)$ such that

$$\mu\bigl(\{x:\ T^n(x)=x \text{ for some } n\}\bigr)=0.$$

Surprisingly, the result for such systems is different from the ergodic case.

**Theorem 4.2.** *Suppose $T$ is an aperiodic measure-preserving automorphism of the Lebesgue space $(X,\mathfrak{B},\mu)$. Let $\mu=\int\mu_t\,dm(t)$ be the decomposition of $\mu$ into ergodic components, and let*

$$h_*(T,\mu) = \text{m-essinf}\{h(T,\mu_t)\} := \sup\Bigl\{c:\ m\{t: h(T,\mu_t)<c\}=0\ \Bigr\}$$

*Then the Rényi entropies are as follows*

$$h(T, q, \mu) = \begin{cases} +\infty, & q < 1, \\ h(T, \mu) = \int h(T, \mu_t)\, dm(t), & q = 1, \\ h_*(T, \mu), & q > 1. \end{cases}$$

This result is a bit surprising because of the following: an entropy-based invariant can detect ergodicity. However, we are not aware of any interesting example, where this observation could be useful. The first candidates, which come to mind, are the non-ergodic Markov shifts, i.e., the shifts for which the transition probability matrix $P$ is not irreducible. It is possible in this case (provided $h(T, \mu) > h_*(T, \mu)$, of course) to show the Rényi entropies of order $q > 1$ are strictly smaller than the measure-theoretic entropy, and thus, the system is not ergodic. However, this proof is much more involved than the standard one, and follows the same idea.

The chapter is organized as follows: in the next section we give a formal definition of the Rényi entropies and establish the basic properties; in section 3 we recall facts about the decomposition into ergodic components. We prove a non-ergodic version of Sinai's theorem on Bernoulli factors, and use it for the computation of the Rényi entropies of order $q > 1$ in section 4. In section 5 we develop a notion of independent partitions in Rokhlin–Halmos towers and subsequently prove the statement for $q < 1$. Finally, in the last section, we pose some open questions about the possible application of Rényi entropies and recently introduced entropy convergence rates.

## 4.2 Rényi entropies of measure preserving transformations

The definition of the Rényi entropy of order $q$ of a measure-preserving transformation goes along the lines of the standard definition of the measure-theoretic (Kolmogorov-Sinai) entropy, and consists of 3 steps: the definition of the Rényi entropy of a finite partition, Rényi entropy of an automorphism with respect to a partition, and, finally, after taking the supremum over all finite partitions, the Rényi entropy of an automorphism, which is a metric invariant.

For any $q \in \mathbb{R}$ the entropy of order $q$ of the partition $\xi = \{\Delta_i\}_{i=1}^n$ is the number

$$H_\mu(q, \xi) = \begin{cases} -\dfrac{1}{q-1} \log\Big(\displaystyle\sum_{i=1}^n \mu(\Delta_i)^q\Big), & \text{for } q \neq 1, \\ -\displaystyle\sum_{i=1}^n \mu(\Delta_i) \log \mu(\Delta_i), & \text{for } q = 1, \end{cases} \tag{4.1}$$

with the standard convention $0^q = 0$ for all $q \in \mathbb{R}$ and $0 \log 0 = 0$.

It is easy to check the following monotonicity property

$$H_\mu(q_1, \xi) \leq H_\mu(q_2, \xi) \text{ for any } \xi \text{ and } q_1 \geq q_2.$$

The Rényi entropy of order $q$ with respect to a partition $\xi$ is defined as

$$h(T,\mu,q,\xi) = \liminf_{n\to\infty} \frac{1}{n} H_\mu\Big(q, \xi^{(n)}\Big), \tag{4.2}$$

where $\xi^{(n)} = \xi \vee T^{-1}\xi \vee \ldots \vee T^{-n+1}\xi$ is the partition into sets $\bigcap_{k=0}^{n-1} T^{-k}\Delta_{i_k}$ with $\Delta_{i_k} \in \xi$.

*Remark.* For $q = 1$ it is known (see for example [4]) that the limit in (4.2) exists. The proof of this fact is based on a so-called *subadditivity property* of the Shannon entropy $H(1,\xi)$:

$$H_\mu(1, \xi \vee \eta) \leq H_\mu(1,\xi) + H_\mu(1,\eta)$$

for all partitions $\xi, \eta$. As it was shown by Rényi in [14], the later is not the case for any $q \neq 1$. This creates some additional problems in the treatment of the Rényi entropies. Nevertheless, if $\xi$ and $\eta$ are independent partitions then

$$H_\mu(q, \xi \vee \eta) = H_\mu(q,\xi) + H_\mu(q,\eta)$$

for all $q \in \mathbb{R}$. We will often exploit this fact.

Finally, we define the Rényi entropy of an automorphism $T$ of order $q$ as the number

$$h(T,\mu,q) = \sup_{\xi} h(T,\mu,q,\xi), \tag{4.3}$$

where the supremum is taken over all finite partitions $\xi$ of $X$.

**Proposition 4.3.** *The Rényi entropies have the following properties:*

*1) $h(T,\mu,q) \geq 0$ for all $q$;*

*2) $h(T,\mu,q_1) \geq h(T,\mu,q_2)$ for $q_1 \leq q_2$;*

*3) $h(T,\mu,1) = h(T,\mu)$, where $h(T,\mu)$ is the measure–theoretic (or Kolmogorov–Sinai) entropy.*

*4) $h(T^n,\mu,q) = nh(T,\mu,q)$ for any $q \in \mathbb{R}$ and every $n \geq 0$.*

Properties 1-3 follow easily from the definition of $h(T,\mu,q)$, and 4 has been established in [22].

## 4.3 Decomposition into ergodic components

Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space [4]. For a measurable partition $\xi = \{C_t\}_{t\in\Lambda}$, where $\Lambda$ can be finite, countable or uncountable, we identify $\Lambda$ and the quotient (or, factor) $X/\xi$ – the space, whose points are the elements of $\xi$. The set $\Lambda$ is a Lebesgue space as well: the set $E \subseteq \Lambda$ is measurable if the set $\cup_{t\in E} C_t$ is a measurable subset of $X$, and we obtain a measure $m$ on $\Lambda$ by letting $m(E) = \mu(\cup_{t\in E} C_t)$. A system of measures $\{\mu_t\}$, $t \in \Lambda$, is called *a canonical system of conditional measures belonging to the partition* $\xi = \{C_t\}_{t\in\Lambda}$, if

1) $\mu_t$ is defined on some $\sigma$-algebra $\mathfrak{B}_t$ of subsets of $C_t$, such that $(C_t, \mathfrak{B}_t, \mu_t)$ is a Lebesgue space.

2) for any $A \in \mathfrak{B}$ the set $A \cap C_t$ belongs to $\mathfrak{B}_t$ for $m$-almost all $t$; the function $\mu_t(A \cap C_t)$ is a measurable function of $t$ and

$$\mu(A) = \int \mu_t(A \cap C_t)\, dm(t).$$

Suppose $T : X \to X$ is a measure–preserving automorphism. Then $(X, \mathfrak{B}, \mu)$ can be decomposed into ergodic components of $T$. By this we mean the following: there exists a $T$-invariant measurable partition $\xi = \{C_t\}$ and a canonical system of conditional measures $\{\mu_t\}$ such that for almost all $t$

$$(C_t, \mathfrak{B}_t, \mu_t, T|_{C_t}) \quad \text{is ergodic.}$$

Suppose $\xi = \{C_t\}$ is the decomposition into ergodic components of $(X, \mathfrak{B}, \mu, T)$, then

$$h(T, \mu) = \int h(T, \mu_t)\, dm.$$

Consider the essential infimum and the essential supremum of measure-theoretic entropies of the measures $\mu_t$ from the decomposition into ergodic components:

$$h_*(T, \mu) = \text{m-essinf}\{h(T, \mu_t) |\, t \in \Lambda = X/\xi\} := \sup\big\{c : m(\{t : h(T, \mu_t) < c\}) = 0\big\},$$
$$h^*(T, \mu) = \text{m-esssup}\{h(T, \mu_t) |\, t \in \Lambda = X/\xi\} := \inf\big\{c : m(\{t : h(T, \mu_t) > c\}) = 0\big\}.$$

The quantity $h^*(T, \mu)$, sometimes called the *entropy rate*, has been previously studied in the literature [9, 21, 23] in relation with the existence of finite generators (generating partitions) for non-ergodic systems. A well-known theorem of Krieger [11] states that an ergodic dynamical system with a finite measure-theoretic entropy $h(T, \mu)$ admits a finite generator $\xi$ with $\mathrm{card}(\xi) \le \exp(h(T, \mu)) + 1$. It turns out that for non-ergodic dynamical systems a similar result is true, provided $h^*(T, \mu) < \infty$: a finite generator $\xi$ exists whose cardinality does not exceed $\exp(h^*(T, \mu)) + 1$.

Denote by $\Pi_m = \{P = (P_1, \dots, P_m)\}$ the set of all ordered partitions of $X$ into $m$ sets. For any measure $\mu$ on $(X, \mathfrak{B})$ define the *partition (pseudo-)metric* $\rho_\mu$ on $\Pi_m$ as follows

$$\rho_\mu(P, Q) = \sum_{k=1}^{m} \mu(P_k \vartriangle Q_k), \quad P, Q \in \Pi_m.$$

If $\rho_\mu(P, Q) = 0$ then $P$ and $Q$ agree except on a set of measure 0, and, of course, in this case we say that $P = Q$. The space $(\Pi_m, \rho_\mu)$ is a complete metric space.

For an at most countable ordered partition $P$ of $(X, \mathfrak{B}, \mu)$ the distribution vector of $P$ is given by

$$d(P, \mu) = \big(\mu(P_1), \mu(P_2), \dots\big).$$

Suppose $P$ and $\bar{P}$ are partitions into $m$ sets of $(X, \mathfrak{B}, \mu)$, $(Y, \mathcal{F}, \nu)$ respectively, then the *distribution distance* is

$$|d(P, \mu) - d(\bar{P}, \nu)| := \sum_{k=1}^{m} |\mu(P_k) - \nu(\bar{P}_k)|.$$

Suppose we have a set $\{\mu_t\}_{t \in \Lambda}$ of measures on $(X, \mathfrak{B})$. For every $t \in \Lambda$ consider the metric $\rho_{\mu_t}$ on $\Pi_m$. The following fact will be often used later: there exists a countable set $\tilde{\Pi}_m \subseteq \Pi_m$, which is $\rho_{\mu_t}$-dense in $\Pi$ for almost every $t \in \Lambda$.

The existence of such $\tilde{\Pi}_m$ follows from the fundamental properties of the Lebesgue spaces. By definition, a Lebesgue space $(X, \mathfrak{B}, \mu)$ admits a countable basis $\Gamma = \{B_\alpha\}$. This in particular means that for any measurable set $A \in \mathfrak{B}$ there exists a set $C$ from a minimal $\sigma$-algebra generated by $\Gamma$ such that

$$C \subseteq A \quad \text{and} \quad \mu(A \setminus C) = 0. \tag{4.4}$$

Denote by $\mathfrak{A}$ the countable algebra generated by $\Gamma$, and let

$$\tilde{\Pi}_m = \Big\{ P = (P_1, \dots, P_m) : \; P_i \in \mathfrak{A} \Big\}.$$

Hence $\tilde{\Pi}_m$ is an at most countable collection of ordered partitions into $m$ sets, where elements of these partitions are taken from $\mathfrak{A}$. From (4.4) we conclude that $\tilde{\Pi}_m$ is $\rho_\mu$ dense in $\Pi_m$. Moreover, for almost every $t \in \Lambda$, $\tilde{\Pi}_m$ is $\rho_{\mu_t}$-dense in $\Pi_m$ as well. This is a consequence of the following fact ([15], see also [16]): for almost every $t \in \Lambda$, the countable collection of sets $\Gamma_t = \Gamma \cap C_t$ is a basis in the Lebesgue space $(C_t, \mathfrak{B}_t, \mu_t)$.

## 4.4 Rényi entropies of order $q > 1$

In this section we are going to prove that $h(T, \mu, q) = h_*(T, \mu)$ for every $q > 1$. We start by showing that $h(T, \mu, q) \leq h_*(T, \mu)$.

### 4.4.1 Estimate from above

Suppose that we have two invariant measures $\mu_1$ and $\mu_2$ for an automorphism $T$. We do not assume these measures to be ergodic. Without loss of generality we can assume that

$$h(T, \mu_1) \leq h(T, \mu_2). \tag{4.5}$$

Consider now another invariant measure $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ with $\alpha \in (0, 1)$. The measure-theoretic entropy of $\mu$ is given by (see [5])

$$h(T, \mu) = \alpha h(T, \mu_1) + (1 - \alpha) h(T, \mu_2).$$

Note that due to (4.5) $h(T, \mu) \geq h(T, \mu_1)$. Let $\xi$ be some finite partition. For any $C \in \xi$ one has

$$\mu(C) = \alpha\mu_1(C) + (1 - \alpha)\mu_2(C),$$

and, therefore, $\mu(C)^q \geq \alpha^q \mu_1(C)^q$ for $q > 1$. Hence, for $q > 1$,

$$H_\mu(q,\xi) = -\frac{1}{q-1}\log\Big(\sum_{C\in\xi}\mu(C)^q\Big) \leq -\frac{q}{q-1}\log\alpha - \frac{1}{q-1}\log\Big(\sum_{C\in\xi}\mu_1(C)^q\Big)$$
$$= -\frac{q}{q-1}\log\alpha + H_{\mu_1}(q,\xi).$$

From the above one easily concludes that

$$h(T,\mu,q,\xi) = \liminf_{n\to\infty}\frac{1}{n}H_\mu(q,\xi^{(n)}) \leq \liminf_{n\to\infty}\frac{1}{n}H_{\mu_1}(q,\xi^{(n)}) = h(T,\mu_1,q,\xi).$$

On the other hand, due to the monotonicity of the Rényi entropies with respect to $q$, for $q > 1$ we have

$$h(T,\mu_1,q,\xi) \leq h(T,\mu_1,q) \leq h(T,\mu_1,1) = h(T,\mu_1).$$

Combining the two last inequalities we finally obtain that for any $q > 1$

$$h(T,\mu,q) \leq h(T,\mu_1).$$

Thus we see that the Rényi entropy of a linear combination of two measures does not exceed a minimum of the measure-theoretic entropies of these two measures. It is evident that the above argument goes through in the case of a finite or countable decomposition: $\mu = \sum_k \alpha_k\mu_k$, where $\alpha_k \geq 0$ and $\sum_k \alpha_k = 1$.

Moreover, the above argument can be equally easily generalized to the case of, generally, uncountable decomposition of an invariant measure $\mu$ into ergodic components $\{\mu_t\}$. This is done in the following lemma.

**Lemma 4.4.** *For a measure preserving system $(X,\mathfrak{B},\mu,T)$ one has*

$$h(T,\mu,q) \leq h_*(T,\mu) \tag{4.6}$$

*for every $q > 1$.*

*Proof.* Consider an ergodic decomposition of $(X,\mathfrak{B},\mu,T)$ as in section 4.3. By the definition of $h_*(T,\mu)$ for every $\varepsilon > 0$ the set $E_1 = \{t : h(T,\mu_t) < h_*(T,\mu)+\varepsilon\}$ has a positive $m$-measure. Suppose, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ one has $m(E_1) < 1$.

If such $\varepsilon_0 > 0$ does not exist, then

$$h(T,\mu_t) = h_*(T,\mu) \quad \text{for} \quad m-a.a.\ t.$$

As a result we immediately conclude that $h(T,\mu) = h_*(T,\mu)$, and using the fact that $h(T,\mu) \geq h(T,\mu,q)$ for any $q > 1$ we obtain our claim (4.6).

Assume such $\varepsilon_0 > 0$ exists and chose any $\varepsilon \in (0,\varepsilon_0)$. Since $m(E_1) \in (0,1)$ we can define

$$\mu_1 = \frac{1}{m(E_1)}\int_{E_1}\mu_t\ dm(t), \quad \mu_2 = \frac{1}{1-m(E_1)}\int_{E_1^c}\mu_t\ dm(t).$$

It is clear that $\mu_1$ and $\mu_2$ are invariant probability measures. Moreover, $h(T, \mu_1) \leq h_*(T, \mu) + \varepsilon$. Using the above argument for two measures $\mu_1$ and $\mu_2$ we conclude that for any $q > 1$

$$h(T, \mu, q) \leq h_*(T, \mu) + \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrary small, we obtain the claim (4.6). □

### 4.4.2 Bernoulli factors of non-ergodic systems

Let us recall a definition of a *Bernoulli automorphism.*

**Definition 4.5.** *An automorphism $T$ of a Lebesgue space $(X, \mathfrak{B}, \mu)$ is called Bernoulli, if it is measure-theoretically isomorphic to a Bernoulli shift.*

If $T$ is a Bernoulli automorphism then there exists a partition $P$ of $X$ such that

1) $P$ is generating,

2) $\{T^n P\}_{n \in \mathbb{Z}}$ is a sequence of independent partitions.

Such partition $P$ is called an independent generator for $T$. A well known theorem by Sinai [19] states that for every ergodic automorphism with positive entropy there exists a Bernoulli factor automorphism with entropy $h \leq h(T, \mu)$.

In this subsection we give a proof of a version of Sinai's theorem on Bernoulli factors for non-ergodic systems. This will be used later in the proof of a lower estimate $h(T, \mu, q) \geq h_*(T, \mu)$ for $q > 1$. Sinai's theorem for non-ergodic automorphisms was first obtained in [10] as a corollary of a more general statement. Our proof is based on a slightly different idea: we establish a non-ergodic version of Ornstein's fundamental lemma, the Sinai theorem then follows in the usual manner. We use some of the basic techniques developed in [10].

The following lemma ([12, Lemma 7], [18, Lemma 9.2], [20, Lemma 8.6]) lies in the hart of the proof of the Ornstein fundamental lemma in the ergodic case.

**Lemma 4.6.** *Let $T$ be an ergodic automorphism of a Lebesgue space $(X, \mathfrak{B}, \mu)$, and $\bar{T}$ be a Bernoulli automorphism of $(Y, \mathfrak{F}, \nu)$ with a finite independent generator $\bar{P}$, card$(\bar{P}) = k$. Assume that $h(T, \mu) \geq h(\bar{T}, \nu)$. Then for any $\varepsilon > 0$ there exist $\delta > 0$ such that:*
*for any measurable partition $P$ of $(X, \mathfrak{B})$, card$(P) = k$, satisfying:*

*1)* $|d(P, \mu) - d(\bar{P}, \nu)| < \delta$,
*2)* $0 \leq h(\bar{T}, \nu, \bar{P}) - h(T, \mu, P) < \delta$,

*and for any $\bar{\delta} > 0$ one can find a partition $Q$ of $X$, card$(Q) = k$, such that*

*3)* $|d(Q, \mu) - d(\bar{P}, \nu)| < \bar{\delta}$,
*4)* $0 \leq h(\bar{T}, \nu, \bar{P}) - h(T, \mu, Q) < \bar{\delta}$,
*5)* $\rho_\mu(P, Q) < \varepsilon$.

The meaning of the above lemma is well-known: for any partition $P$, which is a $\delta$–good 'model' for the Bernoulli process, we can find an even better model (i.e., a partition $Q$, which is $\bar{\delta}$-good, $\bar{\delta} < \delta$) $\varepsilon$-close to $P$. For a non-ergodic dynamical system we would like to have a similar result. It is clear that controlling only the distribution $d(P,\mu)$ and the entropy $h(T,\mu,P)$ is not sufficient. One has to have a sufficiently good control over the situation in each ergodic component. This is done in the following lemma.

**Lemma 4.7.** *Suppose $T$ is an automorphism of a Lebesgue space $(X,\mathfrak{B},\mu)$. Let $\bar{T}$ be a Bernoulli automorphism of $(Y,\mathfrak{F},\nu)$ with a finite independent generator $\bar{P}$, card$(\bar{P})=k$. Let $\{C_t,\mu_t\}$ be a decomposition of $(X,\mathfrak{B},\mu,T)$ into ergodic components and $m$ be a corresponding measure on the factor $X/\{C_t\}$. Assume that $h_*(T,\mu)\geq h(\bar{T},\nu)$. Then for any $\varepsilon>0$ there exist $\delta>0$ such that:*
*for any measurable partition $P$, card$(P)=k$, satisfying:*

*1) $|d(P,\mu_t)-d(\bar{P},\nu)|<\delta$ for $m$-a.e. $t$;*
*2) $0\leq h(\bar{T},\nu,\bar{P})-h(T,\mu_t,P)<\delta$ for $m$-a.e. $t$,*

*and any $\bar{\bar{\delta}}>0$ one can find partition $Q$, card$(Q)=k$, such that*

*3) $|d(Q,\mu_t)-d(\bar{P},\nu)|<\bar{\delta}$ for $m$-a.e. $t$;*
*4) $0\leq h(\bar{T},\nu,\bar{P})-h(T,\mu_t,Q)<\bar{\delta}$ for $m$-a.e. $t$;*
*5) $\rho_{\mu_t}(P,Q)<2\varepsilon$ for $m$-a.e. $t$.*

*Proof.* Let $\varepsilon>0$ and take $\delta=\delta(\varepsilon)>0$ as in Lemma 4.6, i.e., as in ergodic case. Take any partition $P$ such that 1 and 2 hold for $m$-a.e. $t$. Then for $m$-a.e. $t$, using Lemma 4.6, we can find a partition $Q_t$, card$(Q_t)=k$, satisfying:

*3′)* $|d(Q_t,\mu_t)-d(\bar{P},\nu)|<\bar{\delta}$,
*4′)* $0\leq h(\bar{T},\nu,\bar{P})-h(T,\mu_t,Q_t)<\bar{\delta}$,
*5′)* $\rho_{\mu_t}(P,Q_t)<\varepsilon$.

Now the task is to find a universal (following [10]) partition $Q$ such that $Q$ is $\rho_{\mu_t}$-close to $Q_t$ for $m$-a.e. $t$. Without loss of generality we may assume that

$$\gamma_t=h(\bar{T},\nu,\bar{P})-h(T,\mu_t,Q_t)>0 \quad \text{for } m\text{-a.e. } t\ .$$

If this is not the case, we can achieve this by perturbing $Q_t$ slightly without violating conditions 1 and 2, see e.g. [18]

Let $\Pi$ be the set of all ordered measurable partitions of $X$ into $k$ elements, and $\tilde{\Pi}$ be a countable subset, which is $\rho_{\mu_t}$-dense in $\Pi$ for every $t$. Let $\{P^{(1)},\dots,P^{(n)},\dots\}$ be the numeration of the elements of $\tilde{\Pi}$.

The entropy function $h(T,\mu_t,\cdot):\Pi\to\mathbb{R}_+$ is uniformly continuous in $\rho_{\mu_t}$-metric, hence one can find $\beta_t>0$ such that

$$\rho_{\mu_t}(R,S)<\beta_t \quad\Rightarrow\quad \big|h(T,\mu_t,R)-h(T,\mu_t,S)\big|<\frac{\gamma_t}{2}.$$

for any $R,S\in\Pi$. Without loss of generality we may assume that $\beta_t<\min(\varepsilon,\bar{\delta}/2)$. Since $\tilde{\Pi}$ is $\rho_{\mu_t}$-dense in $\Pi$, there exists $n$ such that $\rho_{\mu_t}(Q_t,P^{(n)})<\beta_t$. Denote by $n(t)$ the minimal $n$ with such property. For every $n\geq 1$ put

$$E_n=\{t:\ n(t)=n\},\quad \text{and}\quad B_n=\bigcup_{t\in E_n}C_t,$$

i.e. $E_n$ is the set of parameters, for which $P^{(n)}$ is a good approximation to $Q_t$, and $B_n$ is the union of all ergodic components corresponding to these parameter values.

Obviously, $\{B_n\}_{n\geq 1}$ are pairwise disjoint and $\cup_{n\geq 1} B_n = X$ (mod 0). Finally, we define our universal partition $Q = \{Q_1, \dots, Q_k\}$ as follows:

$$Q_i = \bigcup_{n\geq 1} \left( P_i^{(n)} \cap B_n \right), \quad i = 1, \dots, k.$$

Let us check that this partition $Q$ satisfies conditions 3–5. First of all, one has

$$\rho_{\mu_t}(Q, Q_t) = \rho_{\mu_t}(P^{(n(t))}, Q_t)) < \beta_t. \tag{4.7}$$

Thus

$$\begin{aligned} |d(Q,\mu_t) - d(\bar{P},\nu)| &\leq |d(Q,\mu_t) - d(Q_t,\mu_t)| + |d(Q_t,\mu_t) - d(\bar{P},\nu)| \\ &\leq \rho_{\mu_t}(Q,Q_t) + \bar{\delta} \leq \beta_t + \bar{\delta} \leq \frac{3}{2}\bar{\delta}. \end{aligned}$$

Using (4.7) again, one has

$$0 < \frac{\gamma_t}{2} \leq h(\bar{T},\nu,\bar{P}) - h(T,\mu_t,Q) \leq \frac{3\gamma_t}{2} \leq \frac{3\bar{\delta}}{2},$$

and finally

$$\rho_{\mu_t}(P,Q) \leq \rho_{\mu_t}(P,Q_t) + \rho_{\mu_t}(Q_t,Q) \leq \varepsilon + \beta_t \leq 2\varepsilon.$$

This finishes the proof. □

The usual deduction of Sinai's theorem on Bernoulli factors in the ergodic case consists of applying Lemma 4.6 countably many times. However, one has to prove that the iteration procedure could be started, i.e., for any $\delta > 0$ one can find a partition $P$ such that 1 and 2 hold. For the non-ergodic case this is done in the following lemma.

**Lemma 4.8.** *Suppose $T$ is an automorphism of a Lebesgue space $(X, \mathfrak{B}, \mu)$. Let $\bar{T}$ be a Bernoulli automorphism of $(Y, \mathfrak{F}, \nu)$ with a finite independent generator $\bar{P}$, card$(\bar{P}) = k$. Let $\{C_t, \mu_t\}$ be a decomposition of $(X, \mathfrak{B}, \mu, T)$ into ergodic components and $m$ be the corresponding measure on the factor space $X/\{C_t\}$. Assume that $h_*(T,\mu) \geq h(\bar{T},\nu)$ Then for any $\delta > 0$ there exists a partition $P$, card$(\bar{P}) = k$, such that*

*1) $|d(P,\mu_t) - d(\bar{P},\nu)| < \delta$ for $m$-a.e. $t$;*

*2) $0 \leq h(\bar{T},\nu,\bar{P}) - h(T,\mu_t,P) < \delta$ for $m$-a.e. $t$.*

*Proof.* The proof of this lemma is similar to the proof of lemma 4.7. Since $\mu_t$ is ergodic and $h(T,\mu_t) \geq h(\bar{T},\nu)$ for $m$-a.a $t$, using the results of [12, Lemma 4], [18, Lemma 9.1], we find a partition $P_t$ satisfying

*1′)* $|d(P_t,\mu_t) - d(\bar{P},\nu)| < \delta/2$,

*2′)* $0 \leq h(\bar{T},\nu,\bar{P}) - h(T,\mu_t,P_t) < \delta/2$.

Acting similarly to the proof of Lemma 4.7, we construct a universal partition $P$, which is $\rho_{\mu_t}$ sufficiently close to $P_t$ for $m$-a.e. $t$. As a result we obtain a partition satisfying 1 and 2. $\square$

By now we have collected all the necessary ingredients for the proof of a non-ergodic version of Ornstein's fundamental lemma.

**Theorem 4.9.** *Suppose $T$ is an automorphism of a Lebesgue space $(X, \mathfrak{B}, \mu)$. Let $\bar{T}$ be a Bernoulli automorphism of $(Y, \mathfrak{F}, \nu)$ with a finite independent generator $\bar{P}$, card$(\bar{P}) = k$. Let $\{C_t, \mu_t\}$ be a decomposition of $(X, \mathfrak{B}, \mu, T)$ into ergodic components and $m$ be the corresponding measure on the factor space $X/\{C_t\}$. Assume that $h_*(T, \mu) \geq h(\bar{T}, \nu)$. Then for any $\varepsilon > 0$ there exist $\delta > 0$ such that for any measurable partition $P$, card$(P) = k$, with*

*i)* $|d(P, \mu_t) - d(\bar{P}, \nu)| \leq \delta$ *for $m$-a.e. $t$,*

*ii)* $0 \leq h(\bar{T}, \nu, \bar{P}) - h(T, \mu_t, P) \leq \delta$ *for $m$-a.e. $t$,*

*there exists a partition $Q$, card$(Q) = k$, satisfying:*

*iii)* $\{T^i Q\}$ *is a sequence of independent partitions,*

*iv)* $d(Q, \mu) = d(\bar{P}, \nu)$,

*v)* $\rho_\mu(Q, P) \leq 2\varepsilon$.

*Proof.* The proof of this result is simply a repetition of the corresponding proof in the ergodic case. The only trivial modification, that we have to mention, is the following fact:

$$\rho_\mu(S, R) = \int \rho_{\mu_t}(S, R)\, dm$$

for all partitions $S, R$. $\square$

As a immediate consequence of Ornstein's fundamental lemma for non-ergodic systems we obtain a version of Sinai's theorem on Bernoulli factors.

**Theorem 4.10.** *Suppose $T$ is an automorphism of a Lebesgue space $(X, \mathfrak{B}, \mu)$. Let $\bar{T}$ be a Bernoulli automorphism of $(Y, \mathfrak{F}, \nu)$ with a finite independent generator $\bar{P}$, card$(\bar{P}) = k$. Let $\{C_t, \mu_t\}$ be a decomposition of $(X, \mathfrak{B}, \mu, T)$ into ergodic components and $m$ be a corresponding measure on the factor $X/\{C_t\}$. Assume that $h_*(T, \mu) \geq h(\bar{T}, \nu)$. Then there exists a partition $Q$, card$(\bar{Q}) = k$, such that*

*i)* $\{T^i Q\}$ *is a sequence of independent partitions,*

*ii)* $d(Q, \mu) = d(\bar{P}, \nu)$.

### 4.4.3 Estimate from below

Now we can prove a lower estimate: $h(T,\mu,q) \geq h_*(T,\mu)$ for all $q > 1$. Before we proceed with this estimate we would like to make a couple of remarks. Firstly, we compute the Rényi entropy of order $q$ for a Bernoulli shift. Let $\Omega = \{1,\ldots,k\}^{\mathbb{Z}}$,and $\sigma : \Omega \to \Omega$ be a left shift. Let $\nu$ be a Bernoulli measure on $\Omega$ generated by a probability vector $p = (p_1,\ldots,p_k)$, i.e.,

$$\nu\{\omega = (\omega_i) : \omega_m = a_m,\ldots,\omega_n = a_n\} = p_{a_m}\cdots p_{a_n},$$

for all $m \leq n$ and $a_m,\ldots,a_n \in \{1,\ldots,k\}$. Denote by $\xi$ the partition into the following cylinders:

$$\xi = \{\Delta_1,\ldots,\Delta_k\}, \quad \Delta_n = \{\omega = (\omega_i) : \ \omega_0 = n\}.$$

It is not very difficult to see, that for $q \neq 1$

$$h(\sigma,\nu,q,\xi) = -\frac{1}{q-1}\log\Bigl(\sum_{i=1}^{k} p_i^q\Bigr).$$

In particular, if $p_1 = \ldots = p_k = 1/k$, then $h(\sigma,\nu,q,\xi) = \log k$. In this case, since $h(\sigma,\nu) = \log k$, we immediately conclude that $h(\sigma,\nu,q) = \log k$ for $q > 1$.

Before we continue with a proof of the inequality, we need the following statement.

**Lemma 4.11.** *Suppose $\bar{T} : Y \to Y$ is an automorphism preserving a measure $\nu$. Then for any prime $p \geq 1$ one has*

$$h_*(\bar{T}^p,\nu) = ph_*(\bar{T},\nu).$$

*Proof.* Assume first, that $(\bar{T},\nu)$ is ergodic. If $\bar{T}^p : Y \to Y$ is ergodic, then there is nothing to prove, since in this case $h_*(\bar{T}^p,\nu) = h(\bar{T}^p,\nu) = ph(\bar{T},\nu)$.

If $\bar{T}^p$ is not ergodic, then [17, p.38] there exist disjoint sets $A_0,\ldots,A_{p-1}$, such that $Y = \cup_{i=0}^{p-1} A_i \pmod 0$, $\bar{T}(A_i) = A_{i+1 \mod p}$, and $\bar{T}^p$ is ergodic on $A_0$ with respect to $\nu(\cdot|A_0)$. Therefore, $h_*(\bar{T}^p,\nu) = ph(\bar{T},\nu)$.

Hence, we conclude that if $(\bar{T},\nu)$ is ergodic, then $h_*(\bar{T}^p,\nu) = ph(\bar{T},\nu)$ for any prime $p$, $p \geq 1$.

Assume now that $(T,\mu)$ is not ergodic and let $\mu = \int \mu_t\, dm$ be the decomposition of $\mu$ into ergodic components. Applying the argument above to each $(T,\mu_t)$ we conclude that $h_*(T^p,\mu_t) = ph(T,\mu_t)$, and therefore $h_*(T^p,\mu) = ph_*(T,\mu)$. $\square$

Now let us proceed with the proof of the inequality: $h(T,\mu,q) \geq h_*(T,\mu)$ for all $q > 1$. Assume the opposite, i.e., there exists $q > 1$ such that $h(T,\mu,q) < h_*(T,\mu)$. Take a sufficiently large prime $p$ such that there exists an integer $k$ satisfying

$$ph(T,\mu,q) < \log k < ph_*(T,\mu) = h_*(T^p,\mu). \tag{4.8}$$

Consider a Bernoulli shift, defined as above, with $p_1 = \ldots = p_k = 1/k$. Then by Theorem 4.10 there exists a Bernoulli factor $Q$ for $T^p$ with $\mu(Q_1) = \ldots = \mu(Q_k) = 1/k$. Thus, $h(T^p,\mu,q,Q) = \log k$, but this is in contradiction with (4.8), since $h(T^p,\mu,q) = \sup_R h(T^p,\mu,q,R)$. Hence, $h(T^p,\mu,q) \geq h_*(T,\mu)$ for $q > 1$, and together with (4.6), this gives the equality $h(T^p,\mu,q) = h_*(T,\mu)$ for all $q > 1$.

## 4.5 Rényi entropies of order $q < 1$

In this section we will prove the remaining part of Theorem 4.2. The techniques which we are going to use will be different from the previous section. The reason is that we do not want to assume $h_*(T,\mu) > 0$ (or, even $h(T,\mu) > 0$). In the case when $h_*(T,\mu) > 0$, we can (with the help of the non-ergodic version of Sinai's theorem on Bernoulli factors obtained in the previous section) proceed as in [22].

Our main goal is to construct partitions with arbitrarily large Rényi entropy of order $q$, $q < 1$: for every $C > 0$ we have to find a partition $\xi$ such that

$$h(T,\mu,q,\xi) = \liminf_{k\to\infty} \frac{1}{k} H_\mu(q,\xi^{(k)}) \geq C. \tag{4.9}$$

Since the Rényi entropies are monotonic in $q$ we can restrict ourselves to $q \in (0,1)$.

First of all, let us make an observation which will allow us to simplify the estimate of the Rényi entropy of a partition from below.

**Definition 4.12.** *The Rényi entropy of order $q$, $q \neq 1$, of a finite partition $\eta = \{\Delta_i\}$ restricted to a set $F$, $\mu(F) > 0$, is the number*

$$H_\mu(q,\eta|F) = -\frac{1}{q-1}\log\left(\sum_{\Delta_i\in\eta} \mu(\Delta_i\cap F)^q\right).$$

It is easy to see that any each $q \in (0,1)$ and for any set $F$, $\mu(F) > 0$, one has

$$H_\mu(q,\eta) \geq H_\mu(q,\eta|F). \tag{4.10}$$

In the next subsection we will show how this can be used when $F$ is a base of some Rokhlin–Halmos tower and $\xi$ is some special partition.

### 4.5.1 Rokhlin–Halmos towers and independent collections of sets

We have assumed that $T$ is an aperiodic automorphism. It is well known that for such automorphisms one can construct Rokhlin–Halmos towers of any height and measure arbitrarily close to 1.

Let $M \subseteq X$ then $\tau = \{M, TM, \dots, T^{m-1}M\}$ is called a *Rokhlin–Halmos tower* if

$$T^iM \cap T^jM = \varnothing \text{ for } 0 \leq i \neq j \leq m-1.$$

We will use the same letter $\tau$ for $\cup_{k=0}^{m-1} T^kM$. The height of the tower $\tau$ is said to be $m$ and $\mu(\tau) = m\mu(M)$ is its measure.

We now give a definition of an independent collection of sets relative a (Rokhlin-Halmos) tower. We will associate to such collections certain partitions, which will be analogous to Bernoulli partitions.

**Definition 4.13.** *Let $\tau = \{M, TM, \dots, T^{m-1}M\}$ be a Rokhlin–Halmos tower. We say that a collection $\mathcal{I} = \{I_1, \dots, I_{N-1}\}$ of subsets of $\tau$ is independent relative to $\tau$ if*

*1)* $I_i \cap I_j = \varnothing$ *for* $i \neq j$*;*

*2) denote by* $\xi_{\mathcal{I}}$ *the partition of* $X$ *into the sets* $\{I_1, \ldots, I_{N-1}, I_N := X \setminus \cup_{j=0}^{N-1} I_j\}$*, then*

$$\{T^{-k}(I_1 \cap T^k M), \ldots, T^{-k}(I_N \cap T^k M)\}_{k=0}^{m-1} = \{T^{-k}(\xi_{\mathcal{I}} \cap T^k M)\}_{k=0}^{m-1}$$

*is a collection of independent partitions of* $M$*.*

*For convenience we will always assume that*

*3)* $\mu(I_j \cap T^k M) = \dfrac{\mu(M)}{N}$ *for* $j = 1, \ldots, N$ *and* $k = 0, \ldots, m-1$*.*

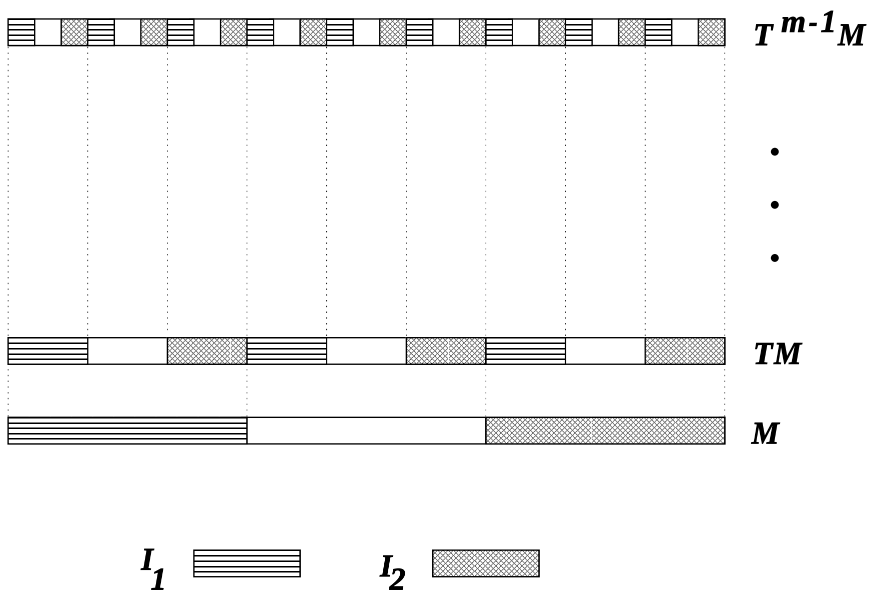


Figure 4.1: Independent sets $(I_1, I_2)$ in $\tau = (M, TM, \ldots, T^{m-1}M)$.

Collections of independent sets exist in every tower. This follows from the following two observations. Firstly, since $T$ is assumed to be aperiodic, the invariant measure $\mu$ has no atoms. Secondly, for any Lebesgue space $(X, \mathfrak{B}, \mu)$, where $\mu$ has no atoms, for every measurable set $A$ and each $\alpha \in [0, \mu(A)]$ one can find a set $B \subseteq A$ with $\mu(B) = \alpha$.

It follows immediately from the definition 4.13, that if $\mathcal{I}$ is a collection of independent sets in $\tau$ and $\xi_{\mathcal{I}}$ is the corresponding partition, then

$$\xi_{\mathcal{I}}^{(k)} \cap M = \left\{ I_{j_1} \cap T^{-1} I_{j_2} \cap \ldots T^{-k+1} I_{j_k} \cap M : \ (j_1, \ldots, j_k) \in \{1, \ldots, N\}^k \right\}$$

is a partition of $M$ into $N^k$ sets of equal measure $\mu(M)/N^k$ for every $k = 1, \dots, m$.

Using (4.10) we easily obtain an estimate on the Rényi entropy of $\xi_{\mathcal{I}}^{(m)}$:

$$\begin{aligned} H_\mu\left(q, \xi_{\mathcal{I}}^{(m)}\right) \geq H_\mu\left(q, \xi^{(m)} | M\right) &= \frac{1}{1-q} \log N^m \left(\frac{\mu(M)}{N^m}\right)^q \\ &= m \log N + \frac{q}{1-q} \log \mu(M). \end{aligned}$$

If the measure of the base of the tower $\mu(M)$ is not too small, say $\mu(M) \geq N^{-m(1-q)/(2q)}$, then $H_\mu(q, \xi_{\mathcal{I}}^{(m)})/m \geq (\log N)/2$.

In the next subsection we estimate the Rényi entropy of a partition which is 'close' to some partition $\xi_{\mathcal{I}}$, where $\mathcal{I}$ is a collection of independent sets.

### 4.5.2 Approximation lemma

Let $\mathcal{I} = (I_1, \dots, I_{N-1})$ be a collection of independent sets in the Rokhlin-Halmos $\tau = (M, \dots, T^{m-1}M)$, and let $\xi = \{I_1, \dots, I_N\}$ be the corresponding partition. Suppose that another partition $\eta = (E_1, \dots, E_N)$ is such that the sets

$$E_j^\tau = E_j \cap \tau$$

are "close" to the corresponding $I_j$'s. Since the partition $\xi^{(m)}$ has a large Rényi entropy (subject to a relation between $N$ and $\mu(\tau)$ of course), then the partition $\eta^{(m)}$ has a large Rényi entropy as well. This can be rigorously formulated in the following way.

**Lemma 4.14.** *Let $\mathcal{I} = \{I_1, \dots, I_{N-1}\}$ be a collection of independent sets in $\tau = (M, \dots, T^{m-1}M)$ with $m \geq 16$, and let $\xi_{\mathcal{I}} = \{I_1, \dots, I_{N-1}, I_N := X \setminus \cup_{j=1}^{N-1} I_j\}$ be the corresponding partition. Suppose $\eta = (E_1, \dots, E_{N-1}, E_N)$ is another partition of $X$ such that*

$$\sum_{j=1}^{N-1} \mu\left(E_j^\tau \triangle I_j\right) \leq \frac{\mu(\tau)}{16(N+31)}. \tag{4.11}$$

*Then for every $q \in (0,1)$ one has*

$$\frac{1}{m} H_\mu(q, \eta^{(m)}) \geq \frac{1}{4} \log N - \log 2 + \frac{q}{(1-q)m} \log \mu(\tau) - \frac{q(1+q)}{2(1-q)m} \log(2m).$$

*Proof.* This lemma is a generalization of lemmas 2.6 and 2.7 from [3], and its proof follows quite closely the proofs of the corresponding results in [3]. Nevertheless, due to the necessary modifications and for the sake of completeness we provide a proof here.

We shall use the following notation: let $\Omega = \{1, \dots, N\}$ and

$$\begin{aligned} \Delta(\mathbf{r}) &= I_{r_1} \cap T^{-1} I_{r_2} \cap \ldots \cap T^{-m+1} I_{r_m} \text{ for } \mathbf{r} = (r_1, \dots, r_m) \in \Omega^m, \\ \tilde{\Delta}(\mathbf{s}) &= E_{s_1} \cap T^{-1} E_{s_2} \cap \ldots \cap T^{-m+1} E_{s_m} \text{ for } \mathbf{s} = (s_1, \dots, s_m) \in \Omega^m. \end{aligned}$$

We know that $\mu(\Delta(\mathbf{r}) \cap M) = \mu(M)/N^m$. Since $\eta$ is close to $\xi_{\mathcal{I}}$ in $\tau$, we expect the sets $\eta^{(m)} \cap M$ to have approximately the same measure as the sets of $\xi^{(m)} \cap M$. Let us make it precise. We say that $\tilde{\Delta}(\mathbf{s})$, $\mathbf{s} \in \Omega^m$, is a 'bad' (or, a 'fat') element of $\eta^{(m)}$ if

$$\mu(\tilde{\Delta}(\mathbf{s}) \cap M) \geq 2^m N^{-m/4} \mu(M),$$

and is 'good' (or 'thin') otherwise. We collect the indexes of all 'bad' elements into the set

$$S = \{\mathbf{s} \in \Omega^m : \tilde{\Delta}(\mathbf{s}) \text{ is 'bad' }\}.$$

We will now show that 'bad' elements of $\eta^{(m)}$ cover less than a half of $M$ in measure, i.e.,

$$\mu\left(\bigcup_{\mathbf{s} \in S} \tilde{\Delta}(\mathbf{s}) \cap M\right) \leq \frac{1}{2}\mu(M).$$

We introduce the following notation:

$$\begin{aligned}
M(\mathbf{s}, \mathbf{r}) &= \tilde{\Delta}(\mathbf{s}) \cap \Delta(\mathbf{r}) \cap M, \\
\tau(\mathbf{s}, \mathbf{r}) &= \bigcup_{k=0}^{m-1} T^k M(\mathbf{s}, \mathbf{r}), \\
\tau(\mathbf{s}) &= \bigcup_{\mathbf{r} \in \Omega^m} \tau(\mathbf{s}, \mathbf{r}).
\end{aligned}$$

It is easy to see that,

$$\begin{aligned}
M(\mathbf{s}, \mathbf{r}) \cap M(\mathbf{s}, \mathbf{t}) = \varnothing \quad &\text{for} \quad \mathbf{r} \neq \mathbf{t}, \\
T^i M(\mathbf{s}, \mathbf{r}) \cap T^j M(\mathbf{s}, \mathbf{r}) = \varnothing \quad &\text{for} \quad i \neq j, \\
\tau(\mathbf{s}, \mathbf{r}) \cap \tau(\mathbf{s}, \mathbf{t}) = \varnothing \quad &\text{for} \quad \mathbf{r} \neq \mathbf{t}.
\end{aligned}$$

Let $\mathbf{s} \in \Omega^m$, then

$$\begin{aligned}
\mu\Big(\big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\big) \cap \tau(\mathbf{s})\Big) &= \sum_{\mathbf{r} \in \Omega^m} \mu\Big(\big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\big) \cap \tau(\mathbf{s}, \mathbf{r})\Big) \\
&= \sum_{\mathbf{r} \in \Omega^m} \sum_{k=0}^{m-1} \mu\Big(\big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\big) \cap T^k M(\mathbf{s}, \mathbf{r})\Big)
\end{aligned} \tag{4.12}$$

Consider the sets participating in the last sum separately. We claim that

$$\Big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\Big) \cap T^k M(\mathbf{s}, \mathbf{r}) = \begin{cases} T^k M(\mathbf{s}, \mathbf{r}), & \text{if } s_k \neq r_k, \\ \varnothing, & \text{if } s_k = r_k. \end{cases} \tag{4.13}$$

For this it is sufficient to show that

$$\Big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\Big) \cap E_{s_k} \cap I_{r_k} = \begin{cases} E_{s_k} \cap I_{r_k} & \text{if } s_k \neq r_k, \\ \varnothing, & \text{if } s_k = r_k. \end{cases} \tag{4.14}$$

The proof is straightforward: let $j = 1, \dots, N-1$ and $k = 1, \dots, m$, then

$$\begin{aligned}(E_j^\tau \vartriangle I_j) \cap E_{s_k} \cap I_{r_k} &= \Big((E_j^\tau \setminus I_j) \cap E_{s_k} \cap I_{r_k}\Big) \cup \Big((I_j \setminus E_j^\tau) \cap E_{s_k} \cap I_{r_k}\Big) \\ &=: A \cup B.\end{aligned}$$

Suppose first that $s_k = r_k$. Then for $j = s_k = r_k$ we have

$$A \cup B \subseteq \Big((E_j^\tau \setminus I_j) \cap I_j\Big) \cup \Big((I_j \setminus E_j^\tau) \cap E_j\Big) = \varnothing,$$

since $I_j \subseteq \tau$ for $j = 1, \dots, N-1$.

For $j \neq s_k = r_k$ we have

$$A \cup B \subseteq \Big((E_j^\tau \setminus I_j) \cap E_{s_k}\Big) \cup \Big((I_j \setminus E_j^\tau) \cap I_{r_k}\Big) = \varnothing,$$

since $E_j \cap E_{s_k} = I_j \cap I_{r_k} = \varnothing$.

Now consider the case $s_k \neq r_k$. If $j \neq s_k$ and $j \neq r_k$, then

$$A \cup B \subseteq (E_j \cap E_{s_k}) \cup (I_j \cap I_{r_k}) = \varnothing.$$

If $j \neq s_k$ and $j = r_k$, then $A \subseteq (E_j^\tau \setminus I_j) \cap E_{s_k} = \varnothing$, but

$$B = (I_j \setminus E_j^\tau) \cap E_{s_k} \cap I_{r_k} = (I_{r_k} \cap E_{s_k}),$$

since $E_j^\tau \cap E_{s_k} = \varnothing$.

Similarly, for $j = s_k$ and $j \neq r_k$, we conclude that $B = \varnothing$, but $A = E_{s_k} \cap I_{r_k}$. Hence we proved (4.14), and therefore (4.13).

Using (4.13) and the fact that $T$ is measure–preserving, we can simplify (4.12):

$$\mu\Big(\big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\big) \cap \tau(\mathbf{s})\Big) = \sum_{\mathbf{r}\in\Omega^m} d_H(\mathbf{s},\mathbf{r})\mu(M(\mathbf{s},\mathbf{r})), \tag{4.15}$$

where $d_H(\mathbf{s},\mathbf{r}) = \#\{k : \ s_k \neq r_k\}$ is the *Hamming distance* between $\mathbf{s}$ and $\mathbf{r}$. We rewrite (4.15) in the following form

$$\mu\left(\big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\big) \cap \tau(\mathbf{s})\right) = \sum_{i=0}^{m} i \sum_{\mathbf{r}:\, d_H(\mathbf{s},\mathbf{r})=i} \mu\big(M(\mathbf{s},\mathbf{r})\big). \tag{4.16}$$

Given $\mathbf{s} \in \Omega^m$ the number of $\mathbf{r}$'s such that $d_H(\mathbf{s},\mathbf{r}) = i$ is $C_m^i(N-1)^i$, where $C_m^i$ is the binomial coefficient. Let us introduce the following notation

$$x_i(\mathbf{s}) := \sum_{\mathbf{r}:\, d_H(\mathbf{s},\mathbf{r})=i} \mu(M(\mathbf{s},\mathbf{r})), \quad y_i := \frac{\mu(M)}{N^m} C_m^i (N-1)^i.$$

Note that $\mu(\tilde{\Delta}(\mathbf{s}) \cap M) = \sum_{i=0}^m x_i(\mathbf{s})$.

Since $M(\mathbf{s},\mathbf{r}) \subseteq \Delta(\mathbf{r}) \cap M$ and $\mu(\Delta(\mathbf{r}) \cap M) = \mu(M)/N^m$, for every $i$ one has

$$x_i(\mathbf{s}) = \sum_{\mathbf{r}:d_H(\mathbf{s},\mathbf{r})=i} \mu(M(\mathbf{s},\mathbf{r})) \le \frac{\mu(M)}{N^m}\left[\sum_{\mathbf{r}:d_H(\mathbf{s},\mathbf{r})=i} 1\right] = \frac{\mu(M)}{N^m} C_m^i (N-1)^i = y_i. \tag{4.17}$$

Furthermore, for every $\mathbf{s}$ there exists $k_{\mathbf{s}} \in \{1, \dots, m\}$ such that

$$\sum_{i=0}^{k_{\mathbf{s}}} y_i > \sum_{i=0}^{m} x_i(\mathbf{s}) \ge \sum_{i=0}^{k_{\mathbf{s}}-1} y_i. \tag{4.18}$$

From (4.17) and (4.18) we conclude that $\sum_{i=l}^{m} x_i(\mathbf{s}) \ge \sum_{i=l}^{k_{\mathbf{s}}-1} y_i$ for all $l \ge 0$, and as a result

$$\sum_{i=0}^{m} i x_i(\mathbf{s}) \ge \sum_{i=0}^{k_{\mathbf{s}}-1} i y_i. \tag{4.19}$$

We will show now that if $\mathbf{s} \in S$ then $k_{\mathbf{s}} \ge \left[\dfrac{3m}{4}\right] + 1$. Indeed, if $\mathbf{s} \in S$, then by definition of $S$,

$$\mu(\tilde{\Delta}(\mathbf{s}) \cap M) \ge 2^m N^{-m/4} \mu(M),$$

and from (4.17) we have

$$\frac{1}{N^m} \sum_{i=0}^{k_s} C_m^i (N-1)^i \ge \frac{\sum_{i=0}^m x_i(\mathbf{s})}{\mu(M)} = \frac{\mu(\tilde{\Delta}(\mathbf{s}) \cap M)}{\mu(M)} \ge 2^m N^{-m/4}.$$

However, Lemma 4.16 (see Appendix below) states that

$$\frac{1}{N^m} \sum_{i=0}^{k} C_m^i (N-1)^i < 2^m N^{-m/4}$$

for all $k = 0, 1, \dots, [3m/4]$. Hence, $k_{\mathbf{s}} \ge [3m/4] + 1$.

Now, for all $\mathbf{s} \in S$ we have

$$\begin{aligned}
\mu\Big((\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)) \cap \tau(\mathbf{s})\Big) &= \sum_{i=0}^{m} i x_i(\mathbf{s}) && \text{(by (4.16))} \\
&\ge \sum_{i=0}^{k_{\mathbf{s}}-1} i y_i = \frac{\mu(M)}{N^m} \sum_{i=0}^{k_{\mathbf{s}}-1} i C_m^i (N-1)^i && \text{(by (4.19))} \\
&\ge \frac{\mu(M)}{N^m} \frac{m}{8(N+31)} \sum_{i=0}^{k_{\mathbf{s}}} C_m^i (N-1)^i && \text{(by Lemma 4.16)} \\
&= \frac{m}{8(N+31)} \sum_{i=0}^{k_{\mathbf{s}}} y_i \\
&\ge \frac{m}{8(N+31)} \sum_{i=0}^{m} x_i(\mathbf{s}) && \text{(by (4.18))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu\Big( \cup_{j=1}^{N-1} (E_j^\tau \vartriangle I_j)\Big) &\geq \sum_{\mathbf{s}\in S} \mu\Big(\big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\big) \cap \tau(\mathbf{s})\Big) \\
&\geq \frac{m}{8(N+31)} \sum_{\mathbf{s}\in S}\sum_{i=0}^{m} x_i(\mathbf{s}) = \frac{m}{8(N+31)} \sum_{\mathbf{s}\in S}\sum_{\mathbf{r}\in\Omega^m} \mu(M(\mathbf{s},\mathbf{r})) \\
&= \frac{m}{8(N+31)} \sum_{\mathbf{s}\in S} \mu(\tilde{\Delta}(\mathbf{s})\cap M).
\end{aligned}$$

Therefore, using our assumption (4.11) one has

$$\begin{aligned}
\sum_{\mathbf{s}\in S} \mu(\tilde{\Delta}(\mathbf{s})\cap M) &\leq \frac{8(N+31)}{m}\mu\Big(\cup_{j=1}^{N-1}(E_j^\tau \vartriangle I_j)\Big) \\
&\leq \frac{8(N+31)}{16(N+31)m}\mu(\tau) = \frac{1}{2}\mu(M),
\end{aligned}$$

i.e., 'bad' elements $\tilde{\Delta}(\mathbf{s})$ cover not more than a half of $M$.

Now we can estimate the Rényi entropy of the partition $\eta^{(m)}$:

$$\begin{aligned}
H_\mu(q,\eta^{(m)}) \geq H_\mu(q,\eta^{(m)}|M) &= \frac{1}{1-q}\log\left(\sum_{\mathbf{s}\in\Omega^m} \mu(\tilde{\Delta}(\mathbf{s})\cap M)^q\right) \\
&\geq \frac{1}{1-q}\log\left(\sum_{\mathbf{s}\in\Omega^m\setminus S} \mu(\tilde{\Delta}(\mathbf{s})\cap M)^q\right) \\
&\geq \frac{1}{1-q}\log\left(\sum_{\mathbf{s}\in\Omega^m\setminus S} \frac{\mu(\tilde{\Delta}(\mathbf{s})\cap M)}{\big(2^m N^{-m/4}\mu(M)\big)^{1-q}}\right) \\
&\geq \frac{1}{1-q}\log\left(\frac{1}{2}\frac{\mu(M)}{\big(2^m N^{-m/4}\mu(M)\big)^{1-q}}\right) \\
&= \frac{m}{4}\log N - m\log 2 + \frac{q}{1-q}\log\mu(M) - \frac{1}{1-q}\log 2 \\
&= m\Big(\frac{1}{4}\log N - \log 2 + \frac{q}{(1-q)m}\log\mu(\tau)\Big) - \frac{1}{1-q}\log 2m^q.
\end{aligned}$$

This finishes the proof of Lemma 4.14. □

### 4.5.3 Partitions with large Rényi entropy

Consider $q \in (0,1)$ and take $N \in \mathbb{N}$, $N \geq 16$. For the convenience of notation we put $\delta = \dfrac{(1-q)}{8q}$. Take $R \in \mathbb{N}$ such that

$$N^{\delta R} > 32(N+31)N.$$

We choose a sequence of Rokhlin-Halmos towers $\{\tau_k\}$,

$$\tau_k = (M_k, \dots, T^{m_k-1}M_k)$$

of height $m_k = Rk$ and measure $\mu(\tau_k) = N^{-\delta Rk}$. For each $k$ let

$$\mathcal{I}_k = (I_1(k), \dots, I_{N-1}(k))$$

be a collection of independent sets in $\tau_k$. We define a sequence of collections of pairwise disjoint sets $\mathcal{E}_k = (E_1(k), \dots, E_{N-1}(k))$ as follows, for $j = 1, \dots, N-1$ let

$$\begin{aligned} &E_j(0) = \varnothing, \\ &E_j(k) = \big(E_j(k-1) \setminus \tau_k\big) \cup I_j(k) \quad \text{ for } k = 1, 2, \dots . \end{aligned}$$

For any $j \in \{1, \dots, N-1\}$ the sequence of characteristic functions $\{\chi_{E_j(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence in $L_1(X, \mathfrak{B}, \mu)$. Indeed, we obviously have

$$E_j(k) \bigtriangleup E_j(k-1) \subseteq \tau_k,$$

and hence for $k_1, k_2 \geq K$ we have

$$\mu\Big(E_j(k_1) \bigtriangleup E_j(k_2)\Big) \leq \sum_{k=K}^{+\infty} \mu(\tau_k) \to 0 \text{ as } K \to \infty.$$

From this we conclude that there exists $E_j \in \mathfrak{B}$ such that

$$\chi_{E_j(k)} \to \chi_{E_j} \text{ for } k \to \infty.$$

It follows from the construction that $\mu(E_j \cap E_i) = 0$ for $i \neq j$. Since we can neglect sets of measure zero we may assume that $E_i \cap E_j = \varnothing$ and hence we have a collection $\mathcal{E} = (E_1, \dots, E_{N-1})$ of pairwise disjoint subsets of $X$.

Furthermore, for every $j = 1, \dots, N-1$ and any $L > k$ one has

$$\begin{aligned} \mu\big(E_j^{\tau_k} \bigtriangleup I_j(k)\big) \leq \mu\big(E_j^{\tau_k} \bigtriangleup E_j^{\tau_k}(L)\big) + \sum_{l=k}^{L-1} \mu\big(E_j^{\tau_k}(l+1) \bigtriangleup E_j^{\tau_k}(l)\big) \\ + \mu\big(E_j^{\tau_k}(k) \bigtriangleup I_j(k)\big). \end{aligned}$$

Moreover, since $E_j^{\tau_k}(k) = I_j(k)$, $\mu\big(E_j \bigtriangleup E_j(L)\big) \to 0$ as $L \to \infty$, and $E_j(l+1) \bigtriangleup E_j(l) \subseteq \tau_l$, we conclude that

$$\mu\big(E_j^{\tau_k} \bigtriangleup I_j(k)\big) \leq \sum_{l=k+1}^{\infty} \mu(\tau_l) = \mu(\tau_k)\frac{N^{-\delta R}}{1 - N^{-\delta R}} < \frac{\mu(\tau_k)}{16(N+31)N},$$

and hence

$$\sum_{j=1}^{N-1} \mu(E_j^{\tau_k} \bigtriangleup I_j(k)) < \frac{\mu(\tau_k)}{16(N+31)}.$$

Now let $\eta = \{E_1, \dots, E_{N-1}, E_N\}$, where $E_N = X \setminus \cup_{j=1}^{N-1} E_j$, and applying Lemma 4.14 we conclude that

$$\begin{aligned}\frac{1}{m_k} H_\mu\left(q, \eta^{(m_k)}\right) &\geq \frac{1}{4}\log N + \frac{q}{(1-q)m_k}\log \mu(\tau_k) + o(1)\\ &= \frac{1}{8}\log N + o(1),\end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$.

For any $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that

$$m_k = Rk \leq n < R(k+1) = m_{k+1}.$$

Since $H_\mu(q, \eta^{(n)}) \geq H_\mu(q, \eta^{(m_k)})$, we have

$$\frac{1}{n} H_\mu\left(q, \eta^{(n)}\right) \geq \frac{m_k}{m_{k+1}} \frac{1}{m_k} H_\mu\left(q, \eta^{(m_k)}\right) \geq \frac{1}{1+\frac{1}{k}}\left(\frac{1}{8}\log N + o(1)\right).$$

This proves that

$$h_\mu(T, q, \eta) = \liminf_{n\to\infty} \frac{1}{n} H_\mu\left(q, \eta^{(n)}\right) \geq \frac{1}{9}\log N.$$

Everywhere above we have assumed that $q \in (0,1)$. However, since

$$h(T, \mu, q, \xi) \geq h(T, \mu, 1/2, \xi)$$

for all $q \leq 0$ and every partition $\xi$, we have obtained partitions with large Rényi entropies of all orders $q$, $q \leq 0$, as well. Finally, since $N$ is an arbitrary integer, we proved the remaining part of Theorem 4.2.

## 4.6 Final remarks

a) Formally speaking, the pair of metric invariants $(h(T, \mu), h(T, \mu, q))$, $q > 1$, can detect ergodicity: if $h(T, \mu) - h(T, \mu, q) > 0$, then $(T, \mu)$ cannot be ergodic. However, we were not able to find any relevant examples where this could be useful.

In our opinion, an example of a measure-preserving system $(X, \mathfrak{B}, \mu, T)$, where the non-ergodicity can be decided from the positiveness of $h(T, \mu) - h(T, \mu, q)$ would be interesting.

b) The difference between ergodicity and non-ergodicity is less interesting than the difference between ergodicity and weak mixing. As it is well known, weak mixing of $T$ is equivalent to the ergodicity of any direct products of $T$ with an ergodic automorphism $S$. Suppose, $T$ is ergodic, but not weakly mixing. Then there exists an ergodic measure-preserving dynamical system $(Y, \mathfrak{F}, \nu, S)$ such that $(X \times Y, \mathfrak{G}, \mu \times \nu, T \times S)$ is not ergodic. Unfortunately, the Rényi entropies are not able to detect non-ergodicity of such systems: for $q > 1$ one has

$$h(T \times S, \mu \times \nu, q) = h(T, \mu, q) + h(S, \nu, q) = h(T, \mu) + h(S, \nu) = h(T \times S, \mu \times \nu),$$

where the first and the third equalities are standard facts for entropy-like characteristics, and the second equality follows from Theorem 4.1

c) *Entropy convergence rates* were introduced in [3].

Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and $T$ be a measure-preserving automorphism. Suppose that $(X, \mathfrak{B}, \mu, T)$ has zero entropy. Hence, for any finite partition $\xi$ one has

$$h(T, \mu, \xi) = \liminf_{n\to\infty} \frac{1}{n} H_\mu(\xi^{(n)}) = 0.$$

Let $c > 0$ and $a_n$, $n \geq 1$, is a sequence of positive numbers such that $a_n \to \infty$. Denote by $\Pi$ the set of all non-trivial partitions $(X, \mathfrak{B}, \mu)$ into two sets.

The automorphism $T$ is said to be

- of type $(LI \geq c)$ for $((a_n), \Pi)$ if for every $\xi \in \Pi$

$$\liminf_{n\to\infty} \frac{1}{a_n} H(\xi^{(n)}) \geq c;$$

- of type $(LS \geq c)$ for $((a_n), \Pi)$ if for every $\xi \in \Pi$

$$\limsup_{n\to\infty} \frac{1}{a_n} H(\xi^{(n)}) \geq c.$$

Similarly one defines types $(LI \leq c)$, $(LS \leq c)$, $(LI < c)$, etc. Clearly, the type of a measure-preserving transformation is a measure-theoretic invariant.

It was shown in [3] that there are no aperiodic transformations of type $(LI < \infty)$ for $((a_n), \Pi)$, where $a_n = o(n)$, $n \geq 1$. Every totally ergodic transformation (i.e., $T^k$ is ergodic for every $k \geq 1$) is of type $(LS = \infty)$ for $(g(\log n), \Pi)$, where $g\colon [0, +\infty) \to \mathbf{R}$ is positive, monotone increasing and

$$\int_1^\infty \frac{g(x)}{x^2} dx < \infty.$$

Also, in [2] F.Blume constructed a class of weakly mixing systems, which can be distinguished by these invariants.

It would interesting to know if the corresponding notions for the Rényi entropies, both in the case of $q < 1$ and $q > 1$, can produce useful convergence rates, which are different from the case of standard entropy.

# Appendix. Auxiliary results

Throughout this section we assume that $N \in \mathbb{N}$, $N \geq 2$, $[x]$ denotes an integer part of $x$, and $C_m^k$ will denote the binomial coefficient

$$C_m^k = \frac{m!}{k!(m-k)!}.$$

We start by deriving simple generalizations of Lemmas 2.3 and 2.4 from [3]. Let $m$ be an integer, for $k=0,\dots,m$ denote

$$a_m(k,N)=\sum_{i=0}^{k} C_m^i\,(N-1)^i,\quad b_m(k,N)=\sum_{i=0}^{k-1} iC_m^i\,(N-1)^i.$$

**Lemma 4.15.** *For $k=1,\dots,m-1$ the following holds:*

$$\frac{a_m(k+1,N)}{b_m(k+1,N)}\le\frac{a_m(k,N)}{b_m(k,N)}.$$

*Proof.* The proof is quite straightforward,

$$\begin{aligned}
&a_m(k,N)b_m(k+1,N)-a_m(k+1,N)b_m(k,N)\\
&=\Big(\sum_{i=0}^{k} C_m^i\,(N-1)^i\Big)kC_m^k(N-1)^k-\Big(\sum_{j=0}^{k-1} jC_m^j\,(N-1)^j\Big)C_m^{k+1}(N-1)^{k+1}\\
&=\sum_{i=0}^{k} kC_m^iC_m^k\,(N-1)^{i+k}-\sum_{i=1}^{k}(i-1)C_m^{i-1}C_m^{k+1}\,(N-1)^{i+k}\\
&\ge\sum_{i=1}^{k}\Big(kC_m^iC_m^k-(i-1)C_m^{i-1}C_m^{k+1}\Big)(N-1)^{i+k}\\
&\ge\sum_{i=1}^{k}k\Big(C_m^iC_m^k-C_m^{i-1}C_m^{k+1}\Big)(N-1)^{i+k}\ge 0;
\end{aligned}$$

the last inequality follows from:

$$\begin{aligned}
C_m^iC_m^k-C_m^{i-1}C_m^{k+1}&=C_m^iC_m^k\Big(1-\frac{i}{m-i+1}\frac{m-k}{k+1}\Big)\\
&\ge\frac{(m-i+1)(k+1)-i(m-k)}{(m-i+1)(k+1)}\\
&=\frac{(m+1)(k-i)+1}{(m-i+1)(k+1)}>0
\end{aligned}$$

for $i\le k$. $\square$

**Lemma 4.16.** *Let $m$ be an integer, $m\ge 16$, and put $k_0=\left[\frac{3m}{4}\right]+1$. Then*

*1) for $k=0,1,\dots,k_0-1$ one has*

$$\frac{a_m(k,N)}{N^m}\le 2^mN^{-m/4}. \tag{4.20}$$

*2) for $k=k_0,\dots,m$ one has*

$$\frac{a_m(k,N)}{b_m(k,N)}\le\frac{8(N+31)}{m}. \tag{4.21}$$

*Proof.* 1) Using this, for $k = \left[\frac{3m}{4}\right]$ we obtain

$$a_m(k,N) = \sum_{i=0}^{k} C_m^i \, (N-1)^i < \left(\sum_{i=0}^{k} C_m^i\right) (N-1)^k \leq 2^m N^{3m/4}.$$

2) Since $a_m(k,N)/b_m(k,N)$ is not increasing with $k$ (see Lemma 4.15), it is sufficient to check (4.21) only for $k = k_0$:

$$\frac{a_m(k_0,N)}{b_m(k_0,N)} = \frac{1 + C_m^{k_0} + \sum_{i=1}^{k_0-1} C_m^i \, (N-1)^i}{\sum_{i=1}^{k_0-1} i \, C_m^i \, (N-1)^i} + \frac{C_m^{k_0} \, (N-1)^{k_0} - C_m^{k_0}}{\sum_{i=1}^{k_0-1} i \, C_m^i \, (N-1)^i} =: I_1 + I_2.$$

We start with $I_1$. To estimate $I_1$ from above we will use the following form of the Chebyshev inequality: let $(p_i)$ be a positive and $(u_i), (v_i)$ be two increasing sequences, then

$$\sum p_i \sum p_i u_i v_i \geq \sum p_i u_i \sum p_i v_i,$$

or equivalently (if everything is positive)

$$\frac{\sum p_i}{\sum p_i v_i} \geq \frac{\sum p_i u_i}{\sum p_i u_i v_i}.$$

Now, applying the Chebyshev inequality to sequences $p_i = C_m^i$, $u_i = (N-1)^i$ and $v_i = i$ we obtain:

$$\begin{aligned}
I_1 &= \frac{1 + C_m^{k_0}}{\sum\limits_{i=1}^{k_0-1} i \, C_m^i \, (N-1)^i} + \frac{\sum\limits_{i=1}^{k_0-1} C_m^i \, (N-1)^i}{\sum\limits_{i=1}^{k_0-1} i \, C_m^i \, (N-1)^i} \\
&\leq \frac{1 + C_m^{k_0}}{\sum\limits_{i=1}^{k_0-1} i \, C_m^i} + \frac{\sum\limits_{i=1}^{k_0-1} C_m^i}{\sum\limits_{i=1}^{k_0-1} i \, C_m^i} \qquad \text{(by the Chebyshev inequality)} \\
&= \frac{\sum\limits_{i=0}^{k_0} C_m^i}{\sum\limits_{i=0}^{k_0-1} i \, C_m^i} \leq \frac{256}{m},
\end{aligned} \tag{4.22}$$

where the last inequality is the result of Lemma 2.4 [3, p.52].

Before we give an upper estimate of $I_2$, we make a simple observation: $C_m^{k_0} \leq C_m^i$ for $i = \left[\frac{m}{4}\right] + 1, \ldots, \left[\frac{3m}{4}\right] = k_0 - 1$. Indeed, if $i \geq m/2$ this is clear since $C_m^i$ is decaying with $i$. For $i < m/2$ we have $C_m^i = C_m^{m-i}$, where $m/2 < m - i \leq k_0$, and hence $C_m^i \geq C_m^{k_0}$. Therefore

$$I_2 \leq \frac{C_m^{k_0}(N-1)^{k_0}}{\sum\limits_{i=1}^{k_0-1} i\, C_m^i\,(N-1)^i} \leq \frac{C_m^{k_0}(N-1)^{k_0}}{\sum\limits_{i=[m/4]+1}^{[3m/4]} i\, C_m^i\,(N-1)^i}$$
$$\leq \frac{C_m^{k_0}(N-1)^{k_0}}{\frac{m}{4} C_m^{k_0} \sum\limits_{i=[m/4]+1}^{[3m/4]} (N-1)^i} = \frac{4(N-1)^{[3m/4]+2}}{m\left(N^{[3m/4]+1} - N^{[m/4]+1}\right)}$$
$$\leq \frac{4(N-1)}{m\left(1 - N^{-[3m/4]+[m/4]}\right)} \leq \frac{8(N-1)}{m} \tag{4.23}$$

since $N \geq 2$ and $m \geq 16$.

Combining together the estimates (4.22) and (4.23), we obtain the inequality (4.21). □

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# Chapter 5

# General multifractal analysis of local entropies

In the present chapter we address the problem of multifractal analysis of local entropies for arbitrary invariant measures. We obtain an upper estimate on the multifractal spectrum of local entropies which is similar to the estimate for local dimensions. We show that in the case of Gibbs measures the above estimate becomes an exact equality. In this case the multifractal spectrum of local entropies is a smooth concave function.

We discuss possible singularities in the multifractal spectrum and their relation to phase transitions.

## 5.1 Introduction

The main problem of multifractal analysis is the description of local singularities of measures. Historically, the multifractal analysis was mainly concerned with the study of local (pointwise) dimensions of a Borel measure $\mu$:

$$d_\mu(x) = \lim_{\varepsilon \to 0} \frac{\log \mu(\mathcal{B}(x, \varepsilon))}{\log \varepsilon},$$

provided the limit exists, where $\mathcal{B}(x, \varepsilon)$ is an open $\varepsilon$-neighborhood of $x$. The purpose is to describe the sets of points with a given pointwise dimension. For this a notion of a multifractal spectrum (sometimes called, a spectrum of singularities) $f(\alpha)$ was introduced in [10]. This spectrum is the following function

$$f(\alpha) = \dim_H(\{x : \ d_\mu(x) = \alpha\}), \tag{5.1}$$

where $\dim_H$ is the Hausdorff dimension.

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This chapter is based on F. Takens, E. Verbitski, *General multifractal analysis of local entropies*, to appear in Fundamenta Mathematicae.

One can generalize this approach even further [2, 17], as we shall explain now. Suppose we are interested in some local (i.e., depending on a point) characteristic of a measure or a dynamical system. We can think of it as a function $F : X \to \mathbb{R}$ ($F$ comes in the place of $d_\mu$). There is a natural multifractal decomposition of the state space $X$ into the level sets of $F$:

$$X = \bigcup_{\alpha\in\mathbb{R}} K_\alpha^F, \tag{5.2}$$

where $K_\alpha^F = \{x \in X : \ F(x) = \alpha\}$. We even may allow the situation, where $F$ is not defined everywhere in $X$. Then we will have to add to the multifractal decomposition (5.2) a set

$$K^F = \{x \in X : \ F(x) \text{ is not defined }\}.$$

The multifractal spectrum $\mathcal{E}$ is by definition the function which assigns "sizes" to the level sets:

$$\mathcal{E}(\alpha) = \mathcal{F}(K_\alpha^F), \quad \alpha \in \mathbb{R},$$

where $\mathcal{F}$ is some set function, i.e., $\mathcal{F}$ is defined on subsets of $X$, and has the property that $\mathcal{F}(Z_1) \le \mathcal{F}(Z_2)$ for $Z_1 \subseteq Z_2$. The set function $\mathcal{F}$ plays the role of the Hausdorff dimension in (5.1).

So such multifractal spectrum is a real function on $\mathbb{R}$ depending on a pair $(F, \mathcal{F})$ as introduced above. Therefore, the multifractal spectrum defined by (5.1) is the multifractal spectrum for local (pointwise) dimensions and the Hausdorff dimension – abbreviated to the multifractal spectrum for local dimensions.

There are only a few local characteristics relevant from the dynamical point of view: local dimension (discussed above), local entropy, Lyapunov exponents. There is also a choice of the set functions: Hausdorff dimension, packing dimension [7], topological entropy for non-compact sets [4, 17]. All these multifractal spectra provide a description of various aspects of dynamical systems (chaoticity, sensitive dependence, etc). These spectra are invariant under smooth conjugations (even homeomorphism with bounded distortion). This is very important in relation to the notion of *multifractal rigidity* introduced in [2].

In the present chapter we study a multifractal spectrum for local entropies. Consider a topological dynamical system $(X, f, \mu)$, where $X$ is a compact metric space, and $f : X \to X$ is a continuous map preserving a Borel probability measure $\mu$. Define the local entropy at a point $x$ as follows:

$$h_\mu(f, x) = \lim_{\varepsilon\to 0} \lim_{n\to 0} -\frac{1}{n} \log \mu(\mathcal{B}_n(x, \varepsilon)),$$

where $\mathcal{B}_n(x,\varepsilon) = \{y \in X : \ d(f^i(x), f^i(y)) < \varepsilon \text{ for } i = 0, \dots, n-1\}$, $\varepsilon > 0$. We define the multifractal spectrum for local entropies and the topological entropy of non-compact sets as

$$\mathcal{E}(\alpha) = h_{top}(f, \{x : \ h_\mu(f, x) = \alpha\}),$$

where $h_{top}(f,Z)$ is the topological entropy of $Z$. We abbreviate this to the spectrum for local entropies. The precise definition is given below, but for a time being one could think of $h_{top}(f,Z)$ as a dynamical analogue of the Hausdorff dimension.

The two multifractal spectra, namely, for local dimensions and for local entropies, contain complementary information about the invariant measure $\mu$ and the dynamical system $f$, as one can see in the example of a skew tent map: for $p \in (0,1)$, put

$$f_p(x) = \begin{cases} x/p, & \text{for } x \in [0,p), \\ (1-x)/(1-p), & \text{for } x \in [p,1]. \end{cases}$$

For the Lebesgue measure, which is invariant, the spectrum for local dimensions is trivial: $d_\mu(x) = 1$ for all $x \in [0,1]$. However for $p \neq 1/2$, the Lebesgue measure has a nontrivial spectrum for local entropies. This will follow from later considerations.

Apart from the pointwise dimensions and entropies also the global dimensions and entropies are defined. There are only a few known characteristics of such type: Rényi dimensions, Hentschel-Procaccia dimension, correlation entropies. For $q \neq 1$ they are given by the following formulas

1) generalized Rényi dimensions:

$$D_q(\mu) = \frac{1}{q-1} \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \sum_i p_i^q,$$

where $p_i = \mu($ $i$-th cell of a partition by cubes of size $\varepsilon$ $)$;

2) Hentschel-Procaccia dimensions:

$$\tilde{D}_q(\mu) = \frac{1}{q-1} \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \int \mu(\mathcal{B}(x,\varepsilon))^{q-1}\, d\mu(x);$$

3) correlation entropies:

$$H_\mu(f,q) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{(q-1)n} \log \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu.$$

The corresponding definitions for $q = 1$ are obtained by the continuity of the "prelimit" quantities. Under mild assumptions the Renyi and Hentschel-Procaccia dimensions coincide [9, 17].

So we consider the generalized dimensions (entropies) of a measure space $(X,\mu)$ (dynamical system $(X,f,\mu)$) as a function which assigns to every $q \in \mathbb{R}$ the dimension (entropy) $D_q$ $(H_\mu(f,q))$.

The standard ("folklore" according to [16]) result of a multifractal analysis establishes that under some conditions, a multifractal spectrum for dimensions is a smooth concave function on a certain interval and is equal to the Legendre transform of the generalized dimensions, scaled by a linear factor. In this case, we say that the multifractal formalism is valid. However, this is not always the case. There are numerous examples where the multifractal formalism fails.

It has been shown in [16] that even in the case, where the multifractal formalism for local dimensions is not valid, one obtains an upper estimate on the multifractal spectrum by taking a Legendre transform of the generalized dimensions.

In the present chapter we obtain similar results for the multifractal spectrum of local (pointwise) entropies. In our case, the spectrum of correlation entropies plays the role of Hentschel-Procaccia and Renyi spectra of dimensions.

Finally, we present examples of chaotic dynamical systems for which the developed methods give a complete or partial description of the multifractal spectrum of local entropies.

## 5.2 Local entropy, Brin-Katok formula

Consider a compact metric space $(X, d)$. Let $f : X \to X$ be a continuous map and $\mu$ an invariant non-atomic Borel probability measure. Without loss of generality we may assume that $\mu$ is positive on open sets. In this case, we define the lower (upper) local (pointwise) entropies as follows:

$$\underline{h}_\mu(f,x) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \; -\frac{1}{n}\log\, \mu(\mathcal{B}_n(x,\varepsilon)), \tag{5.3}$$

$$\overline{h}_\mu(f,x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \; -\frac{1}{n}\log\, \mu(\mathcal{B}_n(x,\varepsilon)), \tag{5.4}$$

where $\mathcal{B}_n(x,\varepsilon) = \{y \in X : \; d(f^i(x), f^i(y)) < \varepsilon \text{ for } i = 0, \dots, n-1\}$. Note that the limits in $\varepsilon$ exist due to the monotonicity.

We say that the local entropy exists at $x$ if

$$\underline{h}_\mu(f,x) = \overline{h}_\mu(f,x). \tag{5.5}$$

In this case the common value will be denoted by $h_\mu(f,x)$.

The following well-known result establishes the existence of the local entropies.

**Theorem 5.1 (Brin–Katok formula, [6]).** *Let $f : X \to X$ be a continuous map on a compact metric space $(X, d)$ preserving a non-atomic measure $\mu$, then*

*1) for $\mu$-a.e. $x \in X$ the local entropy exists, i.e.,*

$$h_\mu(f,x) = \underline{h}_\mu(f,x) = \overline{h}_\mu(f,x);$$

*2) $h_\mu(f,x)$ is an $f$–invariant function of $x$, and*

$$\int h_\mu(f,x)\, d\mu = h_\mu(f),$$

*where $h_\mu(f)$ is the measure–theoretic entropy of $f$.*

*Remark.* If $\mu$ is ergodic then $h_\mu(f,x) = h_\mu(f)$ for $\mu$-a.e. $x \in X$.

In this chapter we study the multifractal spectrum of local entropies. For every $\alpha \geq 0$ consider the level set of local entropies:

$$K_\alpha = \{x \in X : \ h_\mu(f,x) = \alpha\}.$$

We define the multifractal spectrum of local entropies by

$$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha),$$

where $h_{top}(f, K_\alpha)$ is the topological entropy of $f$ restricted to $K_\alpha$ (or simply the topological entropy of $K_\alpha$). The precise definition and basic facts are given in the next section.

## 5.3 Topological entropy of non-compact sets

Suppose $f : X \to X$ is a continuous map of a compact metric space $(X, d)$. Let $\mathcal{U} = \{U_1, \ldots, U_M\}$ be a finite open cover of $X$. By definition, a string $\mathbf{U}$ is a sequence $U_{i_1} \ldots U_{i_n}$ with $i_k \in \{1, \ldots, M\}$, its length $n$ is denoted by $n(\mathbf{U})$. The collection of all strings of length $n$ is denoted by $W_n(\mathcal{U})$ and $W_{\geq n}(\mathcal{U}) = \bigcup_{k \geq n} W_k(\mathcal{U})$. For each $\mathbf{U} \in W_n(\mathcal{U})$ define the set $X(\mathbf{U})$ by

$$\begin{aligned} X(\mathbf{U}) &= U_{i_1} \cap f^{-1} U_{i_2} \cap \ldots \cap f^{-n+1} U_{i_n} \\ &= \{x \in X : \ f^{k-1} x \in U_{i_k}, \ k = 1, \ldots, n\}. \end{aligned}$$

We say that a collection of stings $\Gamma$ covers a set $Z \subseteq X$ if

$$Z \subseteq \bigcup_{\mathbf{U} \in \Gamma} X(\mathbf{U}).$$

For any real number $s$ and every collection of strings $\Gamma$ we introduce the free energy as follows

$$F(\Gamma, s) = \sum_{\mathbf{U} \in \Gamma} \exp(-n(\mathbf{U}) s).$$

For a given $Z$ consider the infimum of free energies over all collections $\Gamma \subseteq W_{\geq n}$ which cover $Z$:

$$M(Z, \mathcal{U}, s, n) = \inf_{\Gamma \text{ covers } Z} F(\Gamma, s), \tag{5.6}$$

and put

$$M(Z, \mathcal{U}, s) = \lim_{n \to \infty} M(Z, \mathcal{U}, s, n).$$

There exists a unique value $\hat{s}$ such that $M(Z, \mathcal{U}, \cdot)$ jumps from $+\infty$ to 0

$$h(Z, \mathcal{U}) := \hat{s} = \sup\{s : M(Z, \mathcal{U}, s) = +\infty\} = \inf\{s : M(Z, \mathcal{U}, s) = 0\}. \tag{5.7}$$

Finally, one can show [17] that the following limit exists

$$h_{top}(f, Z) := \lim_{\mathrm{diam}(\mathcal{U}) \to 0} h(Z, \mathcal{U}). \tag{5.8}$$

**Definition 5.2.** *The number $h_{top}(f, Z)$ is called the topological entropy of $f$ restricted to the set $Z$, or, simply, the topological entropy of $Z$.*

This definition of topological entropy is similar to the definition of the Hausdorff dimension (the diameters of the covering open sets are substituted by $\exp(-n(\mathbf{U}))$ which can be considered as the "dynamical diameter" of $X(\mathbf{U})$). Indeed, these definitions are particular cases of the so-called Carathéodory construction [17] and therefore have similar properties.

**Theorem 5.3 ([17]).** *The topological entropy as defined above has the following properties:*

*1) $h_{top}(f, Z_1) \leq h_{top}(f, Z_2)$ for any $Z_1 \subseteq Z_2 \subseteq X$;*

*2) $h_{top}(f, Z) = \sup\limits_i h_{top}(f, Z_i)$ where $Z = \bigcup\limits_{i=1}^{\infty} Z_i$, $Z_i \subseteq X$;*

*3) if $\mu$ is an invariant measure such that $\mu(Z) = 1$, then $h_{top}(f, Z) \geq h_\mu(f)$.*

## 5.4 $(q, \mu)$-entropy of non-compact or non-invariant sets and entropy doubling condition

In this section following the ideas of [16] and a formalism from [17] we introduce the entropy-related dimension characteristic $h_\mu(f, q, Z)$ which we call the $(q, \mu)$-entropy of $f$ restricted to $Z$, or simply the $(q, \mu)$-entropy of $Z$, when there is no confusion about the dynamics $f$.

This definition requires a few steps and goes along the lines of the definition of topological entropy from the previous section.

For any at most countable collection $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i, \varepsilon)\}_i$ and any $q, t \in \mathbb{R}$ define the $(q, t)$-free energy of $\mathcal{G}$ by

$$F_\mu(\mathcal{G}, q, t) = \sum_i \mu(B_{n_i}(x_i, \varepsilon))^q \exp(-tn_i).$$

For any given set $Z \subseteq X$, $Z \neq \varnothing$, and numbers $q, t \in \mathbb{R}$, $\varepsilon > 0$, $N \in \mathbb{N}$ put

$$M_\mu^c(Z, q, t, \varepsilon, N) = \inf_{\mathcal{G}} F_\mu(\mathcal{G}, q, t) \tag{5.9}$$

where the infimum is taken over all finite or countable collections $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i, \varepsilon)\}_i$ with $x_i \in Z$ and $n_i \geq N$ such that $Z \subseteq \cup_i \mathcal{B}_{n_i}(x_i, \varepsilon)$. To complete the definition we put

$$M_\mu^c(\varnothing, q, t, \varepsilon, N) = 0$$

for any $q, t, \varepsilon$ and $N$.

The quantities $M_\mu^c(Z, q, t, \varepsilon, N)$ are not decreasing with $N$, hence the following limit exists:

$$M_\mu^c(Z, q, t, \varepsilon) = \lim_{N \to \infty} M_\mu^c(Z, q, t, \varepsilon, N) = \sup_{N > 1} M_\mu^c(Z, q, t, \varepsilon, N).$$

Since we consider covers with the centers in a given set, the quantities $M^c_\mu(Z,q,t,\varepsilon)$ are not necessarily monotonic with respect to the set $Z$. We enforce monotonicity by putting

$$M_\mu(Z,q,t,\varepsilon) = \sup_{Z'\subseteq Z} M^c_\mu(Z',q,t,\varepsilon).$$

**Lemma 5.4.** *For any $t \in \mathbb{R}$ the set function $M_\mu(\cdot,q,t,\varepsilon)$ satisfies the following properties:*

*i)* $M_\mu(\varnothing,q,t,\varepsilon) = 0$;

*ii)* $M_\mu(Z_1,q,t,\varepsilon) \le M_\mu(Z_2,q,t,\varepsilon)$ *for any* $Z_1 \subseteq Z_2$;

*iii)* $M_\mu(\cup_i Z_i,q,t,\varepsilon) \le \sum_i M_\mu(Z_i,q,t,\varepsilon)$ *for any* $Z_i \subseteq X$.

**Remark.** In other words, $M(\cdot,q,t,\varepsilon)$ is an outer measure [17]. The role of $M(Z,q,t,\varepsilon)$ is similar to that of the $t$-dimensional Hausdorff measure in the definition of the Hausdorff dimension.

**Lemma 5.5.** *There exists a critical value $h_\mu(f,q,Z,\varepsilon) \in [-\infty,+\infty]$ such that*

$$M_\mu(Z,q,t,\varepsilon) = \begin{cases} \infty, & t < h_\mu(f,q,Z,\varepsilon),\\ 0, & t > h_\mu(f,q,Z,\varepsilon).\end{cases}$$

*Proof.* The statement is a simple consequence of the following standard observation. If $M_\mu(Z,q,\hat{t},\varepsilon) < \infty$ for some $\hat{t}$, then $M_\mu(Z,q,t,\varepsilon) = 0$ for all $t > \hat{t}$. If $M_\mu(Z,q,\hat{t},\varepsilon) > 0$ for some $\hat{t}$, then $M_\mu(Z,q,t,\varepsilon) = \infty$ for all $t < \hat{t}$. □

**Lemma 5.6.** *The following holds:*

*i)* $h_\mu(f,q,\varnothing,\varepsilon) = -\infty$;

*ii)* $h_\mu(f,q,Z_1,\varepsilon) \le h_\mu(f,q,Z_2,\varepsilon)$ *for* $Z_1 \subseteq Z_2$;

*iii)* $h_\mu(f,q,\cup_i Z_i,\varepsilon) = \sup_i h_\mu(f,q,Z_i,\varepsilon)$, *where* $Z_i \subseteq X$, $i = 1,2,\ldots$.

*Proof.* The first statement follows immediately from the fact that $M_\mu(\varnothing,q,t,\varepsilon) = 0$ for any $q,t$. The second statement is a direct consequence of the corresponding statement in the previous lemma.

Note that from $(ii)$ one concludes that $h_\mu(f,q,\cup_i Z_i,\varepsilon) \ge h_\mu(f,q,Z_i,\varepsilon)$ for every $i$. Hence $h_\mu(f,q,\cup_i Z_i,\varepsilon) \ge \sup_i h_\mu(f,q,Z_i,\varepsilon)$. On the other hand, suppose that $h_\mu(f,q,Z_i,\varepsilon) < t$ for all $i$. Thus $M_\mu(Z_i,q,t,\varepsilon) = 0$ for every $i$ and therefore from the previous lemma one has that $M_\mu(\cup_i Z_i,q,t,\varepsilon) = 0$. This means that $h_\mu(f,q,\cup_i Z_i,\varepsilon) \le t$. Hence $h_\mu(f,q,\cup_i Z_i,\varepsilon) \le \sup_i h_\mu(f,q,Z_i,\varepsilon)$ and the result follows. □

We are interested in the asymptotic behavior of local entropies as $\varepsilon \to 0$.

**Definition 5.7.** *The $(q,\mu)$-entropy of $Z$ is*

$$h_\mu(f,q,Z) = \limsup_{\varepsilon\to 0} h_\mu(f,q,Z,\varepsilon). \tag{5.10}$$

Let us discuss the existence of the limit with respect to $\varepsilon$ in the above definition. If $q \leq 0$ then for $\varepsilon_1 > \varepsilon_2 > 0$ one has $h_\mu(f,q,Z,\varepsilon_1) \geq h_\mu(f,q,Z,\varepsilon_2)$ and the limit in (5.10) exists.

Indeed, let $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i,\varepsilon_2)\}$ be a centered cover of $Z$, then $\mathcal{G}' = \{\mathcal{B}_{n_i}(x_i,\varepsilon_1)\}$ is a cover of $Z$ as well and $F_\mu(\mathcal{G},q,t) \geq F_\mu(\mathcal{G}',q,t)$. Therefore $M_\mu(Z,q,t,\varepsilon_2) \geq M_\mu(Z,q,t,\varepsilon_1)$, and hence the limit in (5.10) exists.

In general, for $q > 0$ there is no monotonicity with respect to $\varepsilon$. However under additional assumption on the invariant measure $\mu$ one obtains the monotonic behavior with respect to $\varepsilon$. We formulate this property in the following definition.

**Definition 5.8.** *We say that an invariant measure $\mu$ satisfies the entropy doubling condition if for every sufficiently small $\varepsilon > 0$ the following holds:*

$$C(\varepsilon) := \sup_n \sup_x \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2))} < \infty. \tag{5.11}$$

*Remark.* The constant $1/2$ in the previous definition is not crucial. It is to easy to see that if a measure $\mu$ has the entropy doubling condition, then

$$\sup_n \sup_x \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/a))} < \infty$$

for any $a > 1$. Indeed, suppose $a < 2^k$ for some integer $k$. Then

$$\frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/a))} \leq \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2^k))} = \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2))} \times \ldots \times \frac{\mu(\mathcal{B}_n(x,\varepsilon/2^{k-1}))}{\mu(\mathcal{B}_n(x,\varepsilon/2^k))},$$

and the result follows.

Now we show the existence of the limit in (5.10) for measures satisfying the entropy doubling condition. Indeed, assume that $\mu$ satisfies the entropy doubling condition for all $\varepsilon \in (0,\varepsilon_0)$. Take some $\varepsilon_1,\varepsilon_2 \in (0,\varepsilon_0)$ and let $a = \varepsilon_1/\varepsilon_2 > 1$. From the previous remark we conclude that there exists $\tilde{C} = \tilde{C}(a) < \infty$ such that

$$\frac{\mu(\mathcal{B}_n(x,\varepsilon_1))}{\mu(\mathcal{B}_n(x,\varepsilon_1/a))} = \frac{\mu(\mathcal{B}_n(x,\varepsilon_1))}{\mu(\mathcal{B}_n(x,\varepsilon_2))} < \tilde{C}(a)$$

for all $n$ and $x \in X$. Now let $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i,\varepsilon_2)\}$ be a centered cover of $Z$, then $\mathcal{G}' = \{\mathcal{B}_{n_i}(x_i,\varepsilon_1)\}$ is a centered cover of $Z$ as well and $F_\mu(\mathcal{G},q,t) \geq \tilde{C}^{-q} F_\mu(\mathcal{G}',q,t)$. Thus, $M_\mu(Z,q,t,\varepsilon_2) \geq \tilde{C}^{-q} M_\mu(Z,q,t,\varepsilon_1)$ and $h_\mu(f,q,Z,\varepsilon_2) \geq h_\mu(f,q,Z,\varepsilon_1)$. Hence the limit in (5.10) exists.

Finally, note that all the statements of Lemma 5.6 remain true for $h_\mu(f,Z,q)$ as well.

## 5.5 Coincidence of the topological entropy and $(0,\mu)$-entropy

We begin by making a simple observation which we will often exploit later.

Let $\mathcal{U}$ be a finite open cover of a compact metric space $(X,d)$. Then [25, p.18] there exists a positive number $\delta(\mathcal{U})$, which is called the Lebesgue number of $\mathcal{U}$, such that for any open set $V \subseteq X$ with $\operatorname{diam}(V) < \delta(\mathcal{U})$ there exists an element $U$ of $\mathcal{U}$ containing $V$.

Suppose now that $\varepsilon_1 < \delta(\mathcal{U})/2$. Then for any $x \in X$ and each $n \in \mathbb{N}$, there exists a string $\mathbf{U}_x = U_{i_1} \dots U_{i_n}$ of length $n$ such that

$$\mathcal{B}_n(x,\varepsilon_1) \subseteq X(\mathbf{U}). \tag{5.12}$$

We can obtain this string by choosing as $U_{i_k}$ an element of $\mathcal{U}$ which contains an open ball $\mathcal{B}(f^{k-1}x,\varepsilon_1)$. This is possible since $\varepsilon_1 < \delta(\mathcal{U})/2$.

For an arbitrary string $\mathbf{U} = U_{i_1} \dots U_{i_n}$ take some $x \in X(\mathbf{U}) = U_{i_1} \cap f^{-1}U_{i_2} \cap \dots \cap f^{-n+1}U_{i_n}$, then

$$X(\mathbf{U}) \subseteq \mathcal{B}_{n(\mathbf{U})}(x,\varepsilon_2) \tag{5.13}$$

for any $\varepsilon_2 > 2\operatorname{diam}(\mathcal{U})$.

The measure $\mu$ does not actually participate in the definition of $h_\mu(f,q,Z)$ in the case of $q=0$. This leads to the following fact.

**Lemma 5.9.** *For any set $Z \subseteq X$ one has $h_\mu(f,0,Z) = h_{top}(f,Z)$.*

*Proof.* If $Z = \varnothing$ then the statement is obvious since both sides are equal to $-\infty$. Suppose now that $Z \neq \varnothing$. We start by showing the inequality $h_{top}(f,Z) \leq h_\mu(f,0,Z)$. Let $\mathcal{U}$ be an arbitrary finite open cover of $X$ and $\varepsilon < \delta(\mathcal{U})/2$. Consider an arbitrary collection $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i,\varepsilon)\}_i$ such that $x_i \in Z$ and $Z \subseteq \cup_i \mathcal{B}_{n_i}(x_i,\varepsilon)$.

For any $i$, using (5.12), we choose some string $\mathbf{U}(i)$ of length $n(\mathbf{U}(i)) = n_i$ such that $\mathcal{B}_{n_i}(x_i,\varepsilon) \subseteq X(\mathbf{U}(i))$. Let $\Gamma_{\mathcal{G}} = \{\mathbf{U}(i)\}$. Obviously, $\Gamma_{\mathcal{G}}$ covers $Z$ and

$$F_\mu(\mathcal{G},0,t) = \sum_i \exp(-tn_i) = \sum_{\mathbf{U}\in\Gamma_{\mathcal{G}}} \exp(-tn(\mathbf{U})) = F(\Gamma,t).$$

Since $\mathcal{G}$ is an arbitrary covering, we conclude that

$$M^c_\mu(Z,0,t,\varepsilon,n) = \inf_{\mathcal{G} \text{ covers } Z} F_\mu(\mathcal{G},0,t) \geq \inf_{\Gamma \text{ covers } Z} F(\Gamma,t) = M(Z,\mathcal{U},t,n).$$

Taking the limits as $n \to \infty$ we conclude that

$$M(Z,\mathcal{U},t) \leq M^c_\mu(Z,0,t,\varepsilon) \leq M_\mu(Z,0,t,\varepsilon).$$

Therefore $h(Z,\mathcal{U}) \leq h_\mu(f,0,Z,\varepsilon)$ for any $\varepsilon < \delta(U)/2$, and hence

$$h_{top}(f,Z) = \lim_{\operatorname{diam}(\mathcal{U})\to 0} h(Z,\mathcal{U}) \limsup_{\varepsilon\to 0} h_\mu(f,0,Z,\varepsilon) = h_\mu(f,0,Z).$$

Let us now prove the opposite inequality. Assume $h_\mu(f,0,Z) - h_{top}(f,Z) > 3\gamma > 0$. Then there exists $\varepsilon > 0$ such that

$$h_\mu(f,0,Z,\varepsilon) - h_{top}(f,Z) > 2\gamma.$$

Since $h_{top}(f, Z) = \lim_{\mathrm{diam}(\mathcal{U})\to 0} h(Z,\mathcal{U})$, there exists a finite open cover $\mathcal{U}$ with a sufficiently small diameter (in particular, $\mathrm{diam}(\mathcal{U}) < \varepsilon/2$) such that

$$h_\mu(f, 0, Z, \varepsilon) - h(Z,\mathcal{U}) > \gamma > 0. \tag{5.14}$$

Let $Z'$ be an arbitrary subset of $Z$ and $\Gamma$ an arbitrary collection of strings covering $Z'$. We may assume that $X(\mathbf{U}) \cap Z' \neq \varnothing$ for any $\mathbf{U} \in \Gamma$. Otherwise we just delete those strings and obtain a smaller collection of strings which is still covering $Z'$.

For any $\mathbf{U} \in \Gamma$ choose an arbitrary $x(\mathbf{U}) \in X(\mathbf{U}) \cap Z'$. Then by (5.13) we get

$$X(\mathbf{U}) \subseteq \mathcal{B}_{n(\mathbf{U})}(x(\mathbf{U}), \varepsilon).$$

Therefore the collection $\mathcal{G} = \{\mathcal{B}_{n(\mathbf{U})}(x(\mathbf{U}), \varepsilon)\}$ is a centered cover of $Z'$. Since their free-energies $F(\Gamma, s)$ and $F_\mu(\mathcal{G}, 0, s)$ are the same, we obtain

$$M^c_\mu(Z', 0, s, \varepsilon) \leq M(Z', \mathcal{U}, s) \leq M(Z, \mathcal{U}, s).$$

The last inequality holds due to the monotonicity of $M(\cdot, \mathcal{U}, s)$ with respect to the first argument. As a result we obtain

$$M_\mu(Z, 0, s, \varepsilon) = \sup_{Z' \subseteq Z} M^c_\mu(Z', 0, s, \varepsilon) \leq M(Z, \mathcal{U}, s).$$

This implies

$$h_\mu(f, 0, Z, \varepsilon) \leq h(Z, \mathcal{U})$$

contradicting (5.14) and hence the assumption $h_\mu(f, 0, Z) > h_{top}(f, Z)$ as well. □

## 5.6 Relation between the topological and $(q, \mu)$-entropies of the level sets $K_\alpha$

In this section we are going to establish that actually for any $\alpha \geq 0$ and every $q \in \mathbb{R}$ one has

$$h_{top}(f, K_\alpha) = q\alpha + h_\mu(f, q, K_\alpha). \tag{5.15}$$

On the intuitive level, one can explain the above equality as follows. Suppose $x \in K_\alpha$, then for sufficiently large $n$ and sufficiently small $\varepsilon > 0$ one has

$$\mu(\mathcal{B}_n(x, \varepsilon))^q \exp(-nt) \approx \exp(-n(q\alpha + t)).$$

Using this observation, for any centered covering $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i, \varepsilon)\}$ of $K_\alpha$ we can construct a collection of strings $\Gamma$ such that

$$F_\mu(\mathcal{G}, q, t) \approx F(\Gamma, q\alpha + t). \tag{5.16}$$

The same is true in the opposite direction: for every collection of strings $\Gamma$, we can choose a center covering $\mathcal{G}$ such that (5.16) holds. This implies that the outer measures $M(K_\alpha, q\alpha+t)$ and $M_\mu(K_\alpha, q, t)$, involved in the definitions of $h_{top}(f, K_\alpha)$ and $h_\mu(f, q, K_\alpha)$, are simultaneously equal to $\infty$ or 0. As a results we obtain the corresponding relation between $h_{top}(f, K_\alpha)$ and $h_\mu(f, q, K_\alpha)$. A rigorous proof of this fact is somewhat similar to the proof of the equality $h_{top}(f, Z) = h_\mu(f, 0, Z)$ for all $Z$ from the previous section. However some additional work is required since for the points with local entropy equal to $\alpha$, the convergence of the corresponding limit to $\alpha$ can be non-uniform. Therefore instead of $K_\alpha$ we will consider a monotonic sequence of subsets of $K_\alpha$ which exhaust $K_\alpha$, and for which the convergence of $-\log\mu(\mathcal{B}_n(x,\varepsilon))/n \to \alpha$ is uniform. Both the topological and $(q,\mu)$-entropies have the following property: for any set $Z$ and any sequence $Z_i \subseteq Z$ such that $Z_i \subseteq Z_{i+1}$ and $Z = \cup_i Z_i$ one has

$$h_{top}(f, Z) = \lim_{i\to\infty} h_{top}(f, Z_i), \quad h_\mu(f, q, Z) = \lim_{i\to\infty} h_\mu(f, q, Z_i).$$

Thus, if we establish some relation (similar to (5.15)) between $h_{top}(f, Z_i)$ and $h_\mu(f, q, Z_i)$, we will obtain similar relation for $h_{top}(f, Z)$ and $h_\mu(f, q, Z)$ as well.

Now we make the above arguments rigorous. Consider $\alpha \geq 0$ and the corresponding level set

$$K_\alpha = \{x \in X : \ h_\mu(f, x) = \alpha\} = \Big\{x \in X : \ \lim_{\varepsilon\to 0}\liminf_{n\to\infty} -\frac{1}{n}\log\mu(\mathcal{B}_n(x,\varepsilon)) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log\mu(\mathcal{B}_n(x,\varepsilon)) = \alpha\Big\}.$$

Choose some monotonic sequence $\varepsilon_M \to 0$ as $M \to \infty$. This sequence will be fixed for the rest of this section. Let $\delta > 0$ and put (we omit the $\delta$-dependence in the notation)

$$K_{\alpha,M} = \{x \in K_\alpha : \ \alpha - \delta < \liminf_{n\to\infty} -\frac{1}{n}\log\mu(\mathcal{B}_n(x,\varepsilon_M))\}.$$

Obviously, $K_{\alpha,M} \subseteq K_{\alpha,M+1}$ and $K_\alpha = \bigcup_M K_{\alpha,M}$.

Note that due to the monotonicity of $-\log\mu(\mathcal{B}_n(x,\varepsilon))$ with $\varepsilon$, for each $x \in K_\alpha$ and every $\varepsilon > 0$ one has

$$\limsup_{n\to\infty} -\frac{1}{n}\log\mu(\mathcal{B}_n(x,\varepsilon)) \leq \alpha.$$

For a fixed $x \in K_{\alpha,M}$ there exists $N_0 = N_0(x, \delta, \varepsilon_M)$ such that

$$\alpha - \delta < -\frac{1}{n}\log\mu(\mathcal{B}_n(x,\varepsilon_M)) < \alpha + \delta$$

for all $n > N_0$. Put

$$K_{\alpha,M,N} = \{x \in K_{\alpha,M} : \ N_0 = N_0(x,\delta,\varepsilon_M) < N\}. \tag{5.17}$$

Again, it is easy to see that $K_{\alpha,M,N} \subseteq K_{\alpha,M,N+1}$ and $K_{\alpha,M} = \bigcup_N K_{\alpha,M,N}$.

Let $\mathcal{U}$ be a finite open cover of $X$. Using the properties of topological entropy we conclude that

$$h(K_\alpha, \mathcal{U}) = \lim_{M\to\infty} \lim_{N\to\infty} h(K_{\alpha,M,N}, \mathcal{U}).$$

The following technical lemma establishes the relation between the "pre-measures" participating in the definitions of $h(K_{\alpha,M,N}, \mathcal{U})$ and $h_\mu(f, q, K_{\alpha,M,N}, q, \varepsilon)$.

**Lemma 5.10.** *Suppose $\mathcal{U}$ is an arbitrary open cover of $X$. Consider $K_{\alpha,M,N}$ for some $M, N \in \mathbb{N}$ such that $\varepsilon_M < \delta(\mathcal{U})/2$, where $\delta(\mathcal{U})$ is the Lebesgue number of $\mathcal{U}$. Then for $s \geq q\alpha + |q|\delta + t$ one has*

$$M(K_{\alpha,M,N}, \mathcal{U}, s) \leq M^c_\mu(K_{\alpha,M,N}, q, t, \varepsilon_M).$$

*Proof.* Suppose $n > N$ and $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i, \varepsilon_M)\}_i$ is an arbitrary cover of $K_{\alpha,M,N}$ with $x_i \in K_{\alpha,M,N}$ such that $n_i \geq n$ for all $i$.

By (5.12), for every $x_i$ there exists some string $\mathbf{U}(i)$ with $n(\mathbf{U}(i)) = n_i$ such that $\mathcal{B}_{n_i}(x_i, \varepsilon_M) \subseteq X(\mathbf{U}(i))$ and hence

$$K_{\alpha,M,N} \subseteq \bigcup_i \mathcal{B}_{n_i}(x_i, \varepsilon_M) \subseteq \bigcup_i X(\mathbf{U}(i)).$$

Therefore the collection of strings $\Gamma_\mathcal{G} = \{\mathbf{U}(i)\}$ covers $K_{\alpha,M,N}$. Since $x_i \in K_{\alpha,M,N}$ for all $i$ and $n_i \geq n > N$, we have

$$\exp(-(\alpha+\delta)n_i) \leq \mu(\mathcal{B}_{n_i}(x_i, \varepsilon_M)) \leq \exp(-(\alpha-\delta)n_i).$$

If $q \geq 0$ then $\mu(\mathcal{B}_{n_i}(x_i, \varepsilon))^q \geq \exp(-q(\alpha+\delta)n_i)$ and

$$\begin{aligned}
\sum_i \mu(\mathcal{B}_{n_i}(x_i, \varepsilon))^q \exp(-tn_i) &\geq \sum_i \exp(-n_i(q\alpha + \delta + t)) \\
&\geq \sum_{\mathbf{U}\in\Gamma_\mathcal{G}} \exp(-n(\mathbf{U})s) \\
&\geq M(K_{\alpha,M,N}, \mathcal{U}, s, n)
\end{aligned}$$

for $s \geq q\alpha + q\delta + t$. Since $\mathcal{G}$ is an arbitrary centered covering, we conclude that

$$M(K_{\alpha,M,N}, \mathcal{U}, s, n) \leq M^c_\mu(K_{\alpha,M,N}, q, t, \varepsilon, n) \quad \text{for all} \quad s \geq q\alpha + q\delta + t.$$

Similarly, if $q \leq 0$ then $\mu(\mathcal{B}_{n_i}(x_i, \varepsilon))^q \geq \exp(-(\alpha-\delta)qn_i)$ and

$$\begin{aligned}
\sum_i \mu(\mathcal{B}_{n_i}(x_i, \varepsilon))^q \exp(-tn_i) &\geq \sum_i \exp(-n_i(q\alpha - q\delta + t)) \\
&\geq \sum_{\mathbf{U}\in\Gamma_\mathcal{G}} \exp(-n(\mathbf{U})s) \\
&\geq M(K_{\alpha,M,N}, \mathcal{U}, s, n)
\end{aligned}$$

for $s \geq q\alpha - q\delta + t$. Again, since $\mathcal{G}$ is an arbitrary centered covering, we conclude that

$$M(K_{\alpha,M,N}, \mathcal{U}, s, n) \leq M^c_\mu(K_{\alpha,M,N}, q, t, \varepsilon, n) \leq M_\mu(K_{\alpha,M,N}, q, t, \varepsilon, n).$$

Taking the limits as $n$ tends to infinity, we obtain the desired result. $\square$

**Lemma 5.11.** *For any $\alpha \geq 0$ and any $q \in \mathbb{R}$ one has*

$$h_{top}(T, K_\alpha) \leq q\alpha + h_\mu(T, q, K_\alpha). \tag{5.18}$$

*Proof.* We may assume that $K_\alpha \neq \varnothing$. Otherwise the statement is obvious, because both sides are equal to $-\infty$.

Let us suppose that the opposite is true. Namely, there exists $q \in \mathbb{R}$ such that

$$\gamma = \frac{1}{4}\Big(h_{top}(f, K_\alpha) - q\alpha - h_\mu(f, q, K_\alpha)\Big) > 0.$$

Since $h_{top}(T, K_\alpha) = \lim_{\mathrm{diam}(\mathcal{U}) \to 0} h(K_\alpha, \mathcal{U})$, one can find a finite open cover $\mathcal{U}$ such that

$$h(K_\alpha, \mathcal{U}) > q\alpha + h_\mu(T, q, K_\alpha) + 3\gamma.$$

Take $\delta = \gamma/(2|q|) > 0$ if $|q| > 0$, and let $\delta$ be an arbitrary positive number if $q = 0$. Consider $K_{\alpha,M,N}$ defined in (5.17). Choose sufficiently large $M, N$ such that the three following conditions are full filed:

$$h(K_{\alpha,M,N}, \mathcal{U}) > q\alpha + h_\mu(f, q, K_\alpha) + 2\gamma, \tag{5.19}$$

$$\varepsilon_M < \frac{\delta(\mathcal{U})}{2} \quad \text{and} \quad h_\mu(f, q, K_\alpha) + \frac{\gamma}{2} > h_\mu(f, q, K_\alpha, \varepsilon_M).$$

It is possible to do so because $h(K_\alpha, \mathcal{U}) = \lim\limits_{M\to\infty} \lim\limits_{N\to\infty} h(K_{\alpha,M,N}, \mathcal{U})$, $\varepsilon_M \to 0$ for $M \to \infty$ and $h_\mu(f, q, K_\alpha) = \limsup\limits_{\varepsilon \to 0} h_\mu(f, q, K_\alpha, \varepsilon)$.

By the definition of $h(K_{\alpha,M,N}, \mathcal{U})$, the inequality (5.19) implies

$$M(K_{\alpha,M,N}, \mathcal{U}, q\alpha + h_\mu(T, q, K_\alpha) + 2\gamma) = +\infty.$$

Using the estimates from Lemma 5.10 applied to $s = q\alpha + h_\mu(f, q, K_\alpha) + \gamma$ and $t = h_\mu(f, q, K_\alpha) + \gamma - |q|\delta$, we conclude that

$$M^c_\mu(K_{\alpha,M,N}, q, h_\mu(f, q, K_\alpha) + \gamma - |q|\delta, \varepsilon_M) = \infty. \tag{5.20}$$

Here we arrive at a contradiction with the assumption above. Indeed, one has

$$\begin{aligned} h_\mu(f, q, K_\alpha) + \gamma - |q|\delta &\geq h_\mu(f, q, K_\alpha) + \gamma/2 \\ &> h_\mu(f, q, K_\alpha, \varepsilon_M) \geq h_\mu(f, q, K_{\alpha,M,N}, \varepsilon_M), \end{aligned}$$

and therefore one must have

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(f, q, K_\alpha) + \gamma - |q|\delta, \varepsilon_M) = 0,$$

This finishes the proof. $\square$

Let us prove an inequality, opposite to (5.18). Fix some integers $M, N$ and let $Z \subseteq K_{\alpha,M,N}$, $Z \neq \varnothing$. Consider an arbitrary finite open cover $\mathcal{U}$ with $\mathrm{diam}(\mathcal{U}) <$

$\varepsilon_M/2$. Choose any $n > N$ and let $\Gamma$ be an arbitrary collection of strings covering $Z$ with

$$n(\Gamma) = \inf_{\mathbf{U}\in\Gamma} n(\mathbf{U}) \geq n > N.$$

Without loss of generality we may assume that $X(\mathbf{U}) \cap Z \neq \varnothing$ for any $\mathbf{U} \in \Gamma$. Let $x(\mathbf{U})$ be an arbitrary point from the intersection $X(\mathbf{U}) \cap Z$. Then by (5.13) one has

$$X(\mathbf{U}) \subseteq \mathcal{B}_{n(\mathbf{U})}(x(\mathbf{U}), \varepsilon_M).$$

Hence the collection $\{\mathcal{B}_{n(\mathbf{U})}(x(\mathbf{U}), \varepsilon_M)\}$ is a centered cover of $Z$. Since $x(\mathbf{U}) \in Z \subseteq K_{\alpha,M,N}$ and $n(\mathbf{U}) > N$, one has

$$\exp(-n(\mathbf{U})(\alpha+\delta)) \leq \mu\left(\mathcal{B}_{n(\mathbf{U})}(x(\mathbf{U}), \varepsilon)\right) \leq \exp(-n(\mathbf{U})(\alpha-\delta)) \tag{5.21}$$

Therefore for $q \geq 0$ one has

$$\begin{aligned} M_\mu^c(Z, q, t, \varepsilon_M, n) &\leq \sum_{\mathbf{U}\in\Gamma} \mu\left(\mathcal{B}_{n(\mathbf{U})}(x(\mathbf{U}), \varepsilon)\right)^q \exp(-n(\mathbf{U})t) \\ &\leq \sum_{\mathbf{U}\in\Gamma} \exp(-n(\mathbf{U})(q\alpha - q\delta + t)) \\ &\leq \sum_{\mathbf{U}\in\Gamma} \exp(-n(\mathbf{U})s) \quad \text{ for all } s \leq q\alpha - q\delta + t. \end{aligned}$$

Since $\Gamma$ is an arbitrary covering of $Z$ by strings of length at least $n$, for $s \leq q\alpha - q\delta + t$ we obtain

$$M_\mu^c(Z, q, t, \varepsilon_M, n) \leq M(Z, \mathcal{U}, s, n).$$

Therefore,

$$M_\mu^c(Z, q, t, \varepsilon_M) \leq M(Z, \mathcal{U}, s) \leq M(K_{\alpha,M,N}, \mathcal{U}, s)$$

and

$$M_\mu(K_{\alpha,M,N}, q, t, \varepsilon_M) \leq M(K_{\alpha,M,N}, \mathcal{U}, s).$$

For $q \leq 0$ from (5.21) we have

$$\mu\left(\mathcal{B}_{n(\mathbf{U})}(x(\mathbf{U}), \varepsilon_M)\right)^q \leq \exp(-n(\mathbf{U})q(\alpha+\delta)).$$

Hence,

$$\begin{aligned} M_\mu^c(Z, q, t, \varepsilon_M, n) &\leq \sum_{\mathbf{U}\in\Gamma} \mu\left(\mathcal{B}_{n(\mathbf{U})}(x(\mathbf{U}), \varepsilon)\right)^q \exp(-n(\mathbf{U})t) \\ &\leq \sum_{\mathbf{U}\in\Gamma} \exp(-n(\mathbf{U})(q\alpha + q\delta + t)) \\ &\leq \sum_{\mathbf{U}\in\Gamma} \exp(-n(\mathbf{U})s) \quad \text{ for } \ s \leq q\alpha + q\delta + t. \end{aligned}$$

Similarly to the case $q \geq 0$, we conclude that

$$M_\mu(K_{\alpha,M,N}, q, t, \varepsilon_M) \leq M(K_{\alpha,M,N}, \mathcal{U}, s)$$

Therefore we proved the following result.

**Lemma 5.12.** *Consider some $\delta > 0$ and the corresponding set $K_{\alpha,M,N}$ for some $M, N$. Let $\mathcal{U}$ be any finite open cover with diam$(\mathcal{U}) < \varepsilon_M/2$. Then for $s \leq q\alpha - |q|\delta + t$ one has*

$$M_\mu(K_{\alpha,M,N}, q, t, \varepsilon_M) \leq M(K_{\alpha,M,N}, \mathcal{U}, s).$$

Using the arguments, similar to the proof of Lemma 5.11, we prove the following result.

**Lemma 5.13.** *For any $\alpha \geq 0$ and any $q \in \mathbb{R}$ one has*

$$h_{top}(f, K_\alpha) \geq q\alpha + h_\mu(f, q, K_\alpha).$$

Combining the results of Lemmas 5.11 and 5.13, we easily obtain

**Corollary 5.14.** *For any $\alpha \geq 0$ and any $q \in \mathbb{R}$ one has*

$$h_{top}(f, K_\alpha) = q\alpha + h_\mu(f, q, K_\alpha).$$

## 5.7 Domain of the multifractal spectrum of local entropies

In this section we will establish the first in a series of inequalities, giving an upper estimate on the multifractal spectrum for local entropies.

For this we need the notion of a *Legendre transform*. Let $g : I \to \mathbb{R}$ be a function defined on some interval $I$ which may be infinite or finite. We define its Legendre transform $g^*$ (sometimes also called a conjugate of $g$ [20]) as a function on an interval $I^*$, by

$$g^*(y) = \inf_{x \in I}(xy + g(x)),$$

where $I^* = \{y : \ g^*(y) \text{ is finite}\}$. The interval $I^*$ is called the *domain* of the Legendre transform $g^*$.

Clearly $g^*(y)$ is a concave function, i.e.,

$$g^*(\lambda y_1 + (1-\lambda)y_2) \geq \lambda g^*(y_1) + (1-\lambda)g^*(y_2), \quad \text{for all} \quad \lambda \in [0,1],\ y_1, y_2 \in I^*.$$

Let us now relate the multifractal spectrum $\mathcal{E}(\alpha)$ to a Legendre transform of some function. In the previous sections we have shown that for any $\alpha$ and $q$ one has

$$h_{top}(f, K_\alpha) = q\alpha + h_\mu(f, q, K_\alpha). \tag{5.22}$$

We have not gained much new information about the multifractal spectrum of local entropies since we are still not able to compute $h_\mu(f, q, K_\alpha)$. However, since $K_\alpha \subseteq X$, using the properties of the $(q, \mu)$-entropy, we conclude that for any $q \in \mathbb{R}$ one has $h_\mu(f, q, K_\alpha) \leq h_\mu(f, q, X)$. We introduce the following notation

$$h(q) = h_\mu(f, q, X).$$

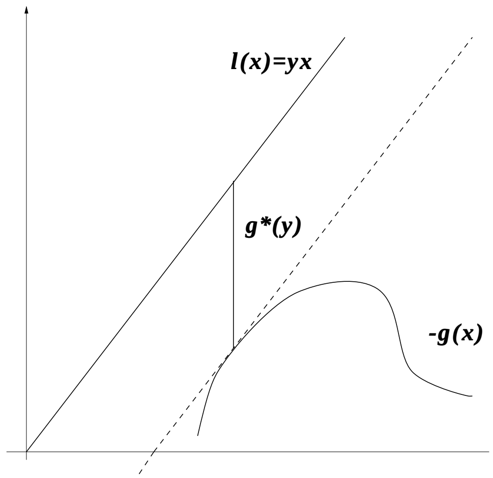


Figure 5.1: Legendre transform: $g^*(y) = \inf_x(yx + g(x)) = \inf_x(yx - (-g(x))$.

Therefore substituting $h(q)$ into (5.22) we conclude

$$h_{top}(f, K_\alpha) \leq q\alpha + h(q) \quad \text{for any } \alpha, q. \tag{5.23}$$

Hence we immediately obtain that

$$\mathcal{E}(\alpha) := h_{top}(f, K_\alpha) \leq h^*(\alpha) := \inf_q \bigl(q\alpha + h(q)\bigr). \tag{5.24}$$

In (5.23) we have deliberately made the estimate of $h_{top}(f, K_\alpha)$ weaker, but this was to relate it to a Legendre transform in (5.24). In the next sections we will relate $h(q)$ to some other characteristics of a dynamical system, namely, the spectrum of correlation entropies.

There is another aspect. Later in the multifractal analysis of Gibbs measures we will show that actually one still has an equality in (5.24). This is due to the fact that for any $\alpha$ with $K_\alpha \neq \varnothing$ there exists a unique $q = q(\alpha)$ such that

$$h_\mu(f, q, K_\alpha) = h_\mu(f, q, X).$$

Therefore despite the fact that we substituted the exact equality (5.22) by the inequality $h_{top}(f, K_\alpha) \leq q\alpha + h(q)$, after taking the infimum over $q$ on the right hand side, we will obtain the equality

$$h_{top}(f, K_\alpha) = h^*(\alpha).$$

One has to stress that this is a rather special case, and, in general, one should not expect equality between the multifractal spectrum $h_{top}(f, K_\alpha)$ and the Legendre transform $h^*(\alpha)$.

In the present section we also study the domain of $h^*(\alpha)$ and show that it contains the domain of a multifractal spectrum of local entropies.

We introduce the following quantities:

$$\underline{\alpha} = \sup_{q>0} -\frac{h(q)}{q}, \quad \overline{\alpha} = \inf_{q<0} -\frac{h(q)}{q},$$

Obviously, $h^*(\alpha) = \inf_q (q\alpha + h(q)) > -\infty$ for $\alpha \in (\underline{\alpha}, \overline{\alpha})$. The next lemma shows that the interval $[\underline{\alpha}, \overline{\alpha}]$ contains the domain of the multifractal spectrum, i.e., $K_\alpha = \varnothing$ (and hence $h_{top}(f, K_\alpha) = -\infty$) if $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$.

**Lemma 5.15.** *The following holds*

$$\underline{\alpha} \leq \inf_{x\in X} \overline{h}_\mu(f,x) \leq \sup_x \underline{h}_\mu(f,x) \leq \overline{\alpha}. \tag{5.25}$$

*Hence $K_\alpha = \varnothing$ for $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$.*

*Proof.* Since there exist points $x \in X$ such that $\underline{h}_\mu(f,x) = \overline{h}_\mu(f,x)$, the middle inequality is obvious.

Suppose now that $\underline{\alpha} > \inf_{x\in X} \overline{h}_\mu(f,x)$. It means that there exist $q_0 > 0$, $x \in X$ and $\delta > 0$ such that

$$-\frac{h(q_0)}{q_0} > \overline{h}_\mu(f,x) + \delta.$$

Hence $t_0 := -q_0(\overline{h}_\mu(f,x) + \delta) > h(q_0)$. Moreover, there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for any $n > N$ one has

$$-\frac{1}{n} \log \mu(\mathcal{B}_n(x,\varepsilon)) \leq \overline{h}_\mu(f,x) + \delta.$$

This implies that for $n > N$ the following holds

$$\mu(\mathcal{B}_n(x,\varepsilon))^{q_0} \exp(-nt_0) \geq \exp\bigl(-nq_0(\overline{h}_\mu(f,x) + \delta) - nt_0\bigr) = 1,$$

where the last equality is due to the choice of $t_0$. Therefore,

$$M_\mu(\{x\}, q_0, t_0, \varepsilon) = \lim_{N\to\infty} \inf_{n\geq N} \mu(\mathcal{B}_n(x,\varepsilon))^{q_0} \exp(-nt_0) \geq 1.$$

Hence,

$$h_\mu(T, q_0, \{x\}) \geq t_0 > h(q_0) = h_\mu(T, q_0, X),$$

which is impossible due to the set monotonicity of $h_\mu(T, q, \cdot)$. Therefore we conclude that $\underline{\alpha} \leq \inf_{x\in X} \overline{h}_\mu(f,x)$.

Assume now that $\overline{\alpha} < \sup_{x\in X} \underline{h}_\mu(f,x)$. Then there exist $q_0 < 0$, $x \in X$ and $\delta > 0$ such that

$$-\frac{h(q_0)}{q_0} < \underline{h}_\mu(f,x) - \delta.$$

Thus $t_0 := -q_0(\underline{h}_\mu(f,x) - \delta) > h(q)$.

Since

$$\underline{h}_\mu(f,x) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x,\varepsilon)),$$

there exist $\varepsilon > 0$ and $N$ such that for any $n > N$

$$-\frac{1}{n} \log \mu(\mathcal{B}_n(x,\varepsilon)) \geq \underline{h}_\mu(f,x) - \delta.$$

Thus, for any $n > N$ (note that $q_0 < 0$)

$$\mu(\mathcal{B}_n(x,\varepsilon))^{q_0} \exp(-nt_0) \geq \exp(-nq_0(\underline{h}_\mu(f,x) - \delta) - nt_0) = 1,$$

due to the choice of $t_0$. As in the previous case, we can conclude that $h(f, q_0, \{x\}) \geq t_0 > h(q_0) = h_\mu(T, q_0, X)$ contradicting the monotonicity of $h(T, q, \cdot)$. Therefore $\sup_x \underline{h}_\mu(f,x) \leq \overline{\alpha}$. □

*Remark.* To be absolutely precise, we have to stress that we actually proved only the following fact:

$$(\underline{\alpha}, \overline{\alpha}) \subseteq \mathrm{dom}(h^*(\alpha)) \quad \text{and} \quad \mathrm{dom}(\mathcal{E}(\alpha)) \subseteq [\underline{\alpha}, \overline{\alpha}].$$

Therefore we can not deduce that $\mathrm{dom}(\mathcal{E}(\alpha)) \subseteq \mathrm{dom}(h^*(\alpha))$. However the difference $\mathrm{dom}(\mathcal{E}(\alpha)) \setminus \mathrm{dom}(h^*(\alpha))$ contains at most 2 points and is irrelevant from the multifractal point of view. More on the situation at the end points can be found in [21].

*Remark.* How far are the estimates $\underline{\alpha}, \overline{\alpha}$ on the domain of the multifractal spectrum from being exact? In the later examples (Gibbs measures), where a complete description of the multifractal spectrum is known, we will see that the above estimates on the domain of the multifractal spectrum are sharp, meaning that not only $K_\alpha = \varnothing$ for $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$, but $K_\alpha \neq \varnothing$ for $\alpha \in (\underline{\alpha}, \overline{\alpha})$.

We summarize the above results in the following statement.

**Theorem 5.16.** *Let $f$ be a continuous transformation of a compact metric space $(X, d)$ with finite topological entropy. Consider an invariant non-atomic Borel measure $\mu$. Then there exist $\underline{\alpha}, \overline{\alpha}$ such that*

*i) $K_\alpha = \varnothing$ for $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$;*

*ii) for $\alpha \in (\underline{\alpha}, \overline{\alpha})$ one has*

$$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) \leq \inf_q (q\alpha + h(q)) = h^*(\alpha),$$

*where $h(q) = h_\mu(f, q, X)$.*

## 5.8 Lower and upper $(q,\mu)$-entropy capacities

The definition of the Hausdorff dimension involves coverings by balls of radius at most $\varepsilon$. If we consider coverings by balls of radius $\varepsilon$ alone we will get the notion of lower and upper capacities. The same idea can be applied to the case of the $(q,\mu)$-entropy $h_\mu(T,q,Z)$. Here, for the set $\mathcal{B}_{n_i}(x_i,\varepsilon)$, the role of the diameter is played by $\exp(-n_i)$. Hence putting $n_i \equiv n$ will be equivalent to fixing the diameters of the covering sets.

Now we give a precise definition of the lower and upper $(q,\mu)$-entropy capacities. Leave the definition of the $(q,t)$-free energy $F_\mu(q,t,\mathcal{G})$ without any changes. Namely, for a collection $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i,\varepsilon)\}_i$ and any $q,t \in \mathbb{R}$, define the $(q,t)$-free energy of $\mathcal{G}$ as

$$F_\mu(\mathcal{G},q,t) = \sum_i \mu(B_{n_i}(x_i,\varepsilon))^q \exp(-tn_i).$$

However, for any $Z \subseteq X$, $Z \neq \varnothing$, we define

$$CM^c_\mu(Z,q,t,\varepsilon,n) = \inf_{\mathcal{G}} F_\mu(q,t,\mathcal{G}), \tag{5.26}$$

where the infimum is taken over all at most countable collection $\mathcal{G} = \{\mathcal{B}_n(x_i,\varepsilon)\}_i$, covering $Z$, with centers $x_i \in Z$. Note that $n$ is the same for all balls in $\mathcal{G}$. Hence the quantities $CM^c_\mu(Z,q,t,\varepsilon,n)$ are not necessarily monotonic in $n$. Therefore define

$$\underline{CM}^c_\mu(Z,q,t,\varepsilon) = \liminf_{n\to\infty} CM^c_\mu(Z,q,t,\varepsilon,n),$$
$$\overline{CM}^c_\mu(Z,q,t,\varepsilon) = \limsup_{n\to\infty} CM^c_\mu(Z,q,t,\varepsilon,n).$$

Again, since we consider only centered coverings, we achieve the set monotonicity by putting

$$\underline{CM}_\mu(Z,q,t,\varepsilon) = \sup_{Z'\subseteq Z} \underline{CM}^c_\mu(Z',q,t,\varepsilon),\ \overline{CM}_\mu(Z,q,t,\varepsilon) = \sup_{Z'\subseteq Z} \overline{CM}^c_\mu(Z',q,t,\varepsilon).$$

There exist unique values $\underline{Ch}_\mu(f,Z,q,\varepsilon)$, $\overline{Ch}_\mu(f,Z,q,\varepsilon) \in [-\infty,+\infty]$ such that

$$\underline{CM}_\mu(Z,q,t,\varepsilon) = \begin{cases} \infty, & \text{if } t < \underline{Ch}_\mu(f,Z,q,\varepsilon), \\ 0, & \text{if } t > \underline{Ch}_\mu(f,Z,q,\varepsilon) \end{cases} \tag{5.27}$$

$$\overline{CM}_\mu(Z,q,t,\varepsilon) = \begin{cases} \infty, & \text{if } t < \overline{Ch}_\mu(f,Z,q,\varepsilon), \\ 0, & \text{if } t > \overline{Ch}_\mu(f,Z,q,\varepsilon). \end{cases} \tag{5.28}$$

Finally, we define the lower and upper capacities as

$$\underline{Ch}_\mu(f,Z,q) = \limsup_{\varepsilon\to 0} \underline{Ch}_\mu(f,Z,q,\varepsilon), \quad \overline{Ch}_\mu(f,Z,q) = \limsup_{\varepsilon\to 0} \overline{Ch}_\mu(f,Z,q,\varepsilon).$$

The lower and upper capacities admit a different equivalent definition. Namely, for every set $Z \subseteq X$, $Z \neq \varnothing$, define

$$\Lambda(Z,q,\varepsilon,n) = \inf_{\mathcal{G}} \sum_i \mu(\mathcal{B}_n(x_i,\varepsilon))^q,$$

where the infimum is taken over all finite or countable collections $\mathcal{G} = \{\mathcal{B}_n(x_i, \varepsilon)\}_i$ with $x_i \in Z$ and such that $Z \subseteq \cup_i \mathcal{B}_n(x_i, \varepsilon)$. By definition we let $\Lambda(\varnothing, q, \varepsilon, n) = 0$.

The next statement (similar to [17, Theorem 2.2]) gives an equivalent definition of $\underline{Ch}_\mu(f, Z, q, \varepsilon)$ and $\overline{Ch}_\mu(f, Z, q, \varepsilon)$.

**Lemma 5.17.** *For any $Z \subseteq X$, $Z \neq \varnothing$, we have*

$$\underline{Ch}_\mu(f, Z, q, \varepsilon) = \sup_{Z' \subseteq Z} \liminf_{n\to\infty} \frac{\log \Lambda(Z', q, \varepsilon, n)}{n},$$

$$\overline{Ch}_\mu(f, Z, q, \varepsilon) = \sup_{Z' \subseteq Z} \limsup_{n\to\infty} \frac{\log \Lambda(Z', q, \varepsilon, n)}{n}.$$

*Proof.* We prove the equality only for the case of lower entropy capacity. The other case is analogous. Denote for simplicity

$$a = \underline{Ch}_\mu(f, Z, q, \varepsilon), \quad b = \sup_{Z' \subseteq Z} \liminf_{n\to\infty} \frac{\Lambda(Z', q, \varepsilon, n)}{n}.$$

Let $\delta > 0$, then by definition (5.27), for every $Z' \subseteq Z$ one has

$$0 = \underline{CM}_\mu(Z', q, a+\delta, \varepsilon) = \liminf_{n\to\infty} \inf_{\mathcal{G}} \sum_i \mu(\mathcal{B}_n(x_i, \varepsilon))^q \exp(-n(a+\delta)).$$

Therefore there exists a sequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$, and a sequence of $Z'$-centered coverings $\mathcal{G}_k = \{\mathcal{B}_{n_k}(x_i, \varepsilon)\}$ such that

$$F_\mu(q, a+\delta, \mathcal{G}_{n_k}) \to 0 \text{ as } k \to \infty.$$

Since $\Lambda(Z', q, \varepsilon, n_k) \exp(-n_k(a+\delta)) \leq F_\mu(q, a+\delta, \mathcal{G}_{n_k})$, we conclude that

$$\liminf_{n\to\infty} \frac{\log \Lambda(Z', q, \varepsilon, n)}{n} \leq a + \delta.$$

Since $Z'$ is an arbitrary subset of $Z$ and $\delta > 0$ is an arbitrary number, we conclude that $b \leq a$.

Now we prove the opposite inequality. For any $\delta > 0$ one has

$$\underline{CM}_\mu(Z, q, a-\delta, \varepsilon) = \sup_{Z' \subseteq Z} \underline{CM}^c_\mu(Z, q, a-\delta, \varepsilon) = \infty.$$

Hence there exists a set $Z' \subseteq Z$ such that

$$\underline{CM}^c_\mu(Z', q, a-\delta, \varepsilon) = \liminf_{n\to\infty} \inf_{\mathcal{G}} \sum_i \mu(\mathcal{B}_n(x_i, \varepsilon))^q \exp(-n(a-\delta)) > 2$$

and therefore for sufficiently large $N$ and all $n > N$ and every centered covering $\mathcal{G}$ of $Z'$ one has

$$F_\mu(q, a-\delta, G) = \Big(\sum_i \mu(\mathcal{B}_n(x_i, \varepsilon))^q\Big) \exp(-n(a-\delta)) > 1.$$

Thus, for all $n > N$

$$\frac{\log \Lambda(Z', q, \varepsilon, n)}{n} \geq a - \delta,$$

This implies that $b \geq \liminf_{n\to\infty} \log \Lambda(Z', q, \varepsilon, n)/n \geq a - \delta$. Since $\delta$ was chosen to be arbitrary, we conclude that $b \geq a$. This completes the proof of the statement.

$\square$

It is obvious that the $(q,\mu)$-entropy and the lower and upper capacities satisfy the following inequality

$$h_\mu(f,q,Z) \leq \underline{Ch}_\mu(f,q,Z) \leq \overline{Ch}_\mu(f,q,Z)$$

for any $q \in \mathbb{R}$ and all $Z \subseteq X$. Therefore we can substitute $\underline{Ch}(q) := \underline{Ch}_\mu(f,q,X)$ and $\overline{Ch}(q) := \overline{Ch}_\mu(f,q,X)$ for $h(q) = h_\mu(f,q,X)$ in the statement of Theorem 5.16 and obtain the following theorem.

**Theorem 5.18.** *Let $f$ be a continuous transformation of a compact metric space $(X,d)$ with finite topological entropy. Consider an invariant non-atomic Borel measure $\mu$. Then there exist $\underline{\alpha}, \overline{\alpha}$ such that*

*i) $K_\alpha = \varnothing$ for $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$;*

*ii) for $\alpha \in (\underline{\alpha}, \overline{\alpha})$ one has*

$$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) \leq \underline{Ch}^*(\alpha) \leq \overline{Ch}^*(\alpha),$$

*where $\underline{Ch}(q) = \underline{Ch}_\mu(f,q,X)$ and $\overline{Ch}(q) = \overline{Ch}_\mu(f,q,X)$.*

## 5.9 Correlation entropies

The spectra of generalized dimensions has been successfully used in the multifractal analysis of local dimensions [17, 16]. Two spectra have been considered: the first, named after Rényi, is based on the box counting ideas, the second (sometimes called the Hentschel-Procaccia spectrum) is based on the correlation integrals. Under mild assumptions these spectra coincide [9, 17].

The generalization to the entropy case were made in [8, 3]. However the first approach produces the Rényi entropies which do not contain any essential new information about the dynamical system [22].

On the other hand, the correlation entropies, which we define in this section, do not have these problems and suit the purposes of the multifractal analysis of local entropies quite well.

Define

$$I(q,\varepsilon,n) = \begin{cases} -\dfrac{1}{q-1}\log\Big(\displaystyle\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu\Big), & q \neq 1, \\ -\displaystyle\int \log \mu(\mathcal{B}_n(x,\varepsilon))\, d\mu, & q = 1. \end{cases}$$

The lower and upper correlation entropies are then defined as

$$\underline{H}_\mu(f,q) = \lim_{\varepsilon\to 0}\liminf_{n\to\infty} \frac{1}{n}\ I(q,\varepsilon,n),$$
$$\overline{H}_\mu(f,q) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\ \log I(q,\varepsilon,n),$$

Applying the Brin-Katok formula and the Fatou lemma, we conclude that actually

$$\underline{H}_\mu(f,1) = \overline{H}_\mu(f,1) = h_\mu(f).$$

One can easily show that $\underline{H}_\mu(f,q)$ and $\overline{H}_\mu(f,q)$ are continuous functions for $q \in [0,1)$ and $q \in (1,\infty)$, with a possible discontinuity at $q=1$ (see the last section of this chapter for the examples).

Moreover, the following (quite rough) estimates are known (see chapter 2)

$$0 \le \underline{H}_\mu(f,q) \le \overline{H}_\mu(f,q) \le h_\mu(f) \text{ for } q>1,$$

$$h_\mu(f) \le \underline{H}_\mu(f,q) \le \overline{H}_\mu(f,q) \le h_{top}(f) \text{ for } q\in[0,1).$$

Since we will later use the correlation entropies in our estimates of the multifractal spectrum, and we would like to keep these estimates as sharp as possible, we will only use the lower correlation entropies. Hence we introduce the following notation

$$H_\mu(f,q) = \underline{H}_\mu(f,q)$$

## 5.10 Entropy doubling condition and correlation entropies

In this section we assume that the invariant measures satisfies the entropy doubling condition (see formula (5.11)). This will allow us to relate the multifractal spectrum for local entropies to the spectrum of correlation entropies, defined in the previous section.

**Lemma 5.19.** *If $\mu$ satisfies the entropy doubling condition then for every sufficiently small $\varepsilon>0$ and any $q>1$ there exist constants $C_1, C_2$ such that for any $Z\subseteq X$ one has*

$$C_1\Lambda(Z,q,2\varepsilon,n) \le \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu \le C_2\Lambda(X,q,\varepsilon,n)$$

*Proof.* Let $Z\subseteq X$ and $E=\{x_i\}$ be some maximal $(n,\varepsilon)$-separated set, i.e., for every $i\ne j$ one has $d_n(x_i,x_j)>\varepsilon$ and one cannot add any other points to $E$ without violating the separation condition. This in particular implies that $\mathcal{B}_n(x_i,\varepsilon/2)\cap\mathcal{B}_n(x_j,\varepsilon/2)=\varnothing$ and $X\subseteq\cup_i\mathcal{B}_n(x_i,\varepsilon)$.

Obviously, for $q > 1$

$$\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu \geq \sum_i \int\limits_{\mathcal{B}_n(x_i,\varepsilon/2)} \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu \geq \sum_i \mu(\mathcal{B}_n(x_i,\varepsilon/2))^q, \tag{5.29}$$

since $\mathcal{B}_n(x_i,\varepsilon/2) \subseteq \mathcal{B}_n(x,\varepsilon)$ for every $x \in \mathcal{B}_n(x_i,\varepsilon/2)$.

On the other hand, for any $i$ such that $\mathcal{B}_n(x_i,\varepsilon) \cap Z \neq \varnothing$ choose some $z_i \in \mathcal{B}_n(x_i,\varepsilon) \cap Z$. Denote the set of all such $i$'s by $I'$. Thus $\mathcal{B}_n(x_i,\varepsilon) \subseteq \mathcal{B}_n(z_i,2\varepsilon)$ and $Z \subseteq \cup_i \mathcal{B}_n(z_i,2\varepsilon)$. Hence $\{\mathcal{B}_n(z_i,2\varepsilon)\}_{i\in I'}$ is a centered covering of $Z$. Note that $\mathcal{B}_n(z_i,\varepsilon) \subseteq \mathcal{B}_n(x_i,2\varepsilon)$ as well.

Let $C = C(\varepsilon) \in (1,\infty)$ be such that

$$\sup_x \frac{\mu(\mathcal{B}_n(x,2\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon))} < C \quad \text{and} \quad \sup_x \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2))} < C \tag{5.30}$$

for all $n$. Such $C$ exists due to the entropy-doubling condition. Hence,

$$\begin{aligned}\Lambda(Z,q,2\varepsilon,n) &\leq \sum_{i\in I'} \mu(\mathcal{B}_n(z_i,2\varepsilon))^q \leq C^q \sum_{i\in I'} \mu(\mathcal{B}_n(z_i,\varepsilon))^q \leq C^q \sum_{i\in I'} \mu(\mathcal{B}_n(x_i,2\varepsilon))^q \\ &\leq C^{2q} \sum_{i\in I} \mu(\mathcal{B}_n(x_i,\varepsilon))^q \leq C^{3q} \sum_{i\in I} \mu(\mathcal{B}_n(x_i,\varepsilon/2))^q.\end{aligned}$$

Therefore, comparing (5.29) with the previous estimate, we conclude that for $q > 1$ and any $Z \subseteq X$

$$\frac{1}{C^{3q}}\Lambda(Z,q,2\varepsilon,n) \leq \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu.$$

Now let us prove the remaining upper estimate. Let $E = \{x_i\}$ be an arbitrary $(n,\varepsilon)$-generating set, i.e., $X = \cup_i \mathcal{B}_n(x_i,\varepsilon)$. Then

$$\begin{aligned}\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu &\leq \sum_i \int_{\mathcal{B}_n(x_i,\varepsilon)} \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu \\ &\leq \sum_i \mu\left(\mathcal{B}_n(x_i,2\varepsilon)\right)^{q-1} \mu(\mathcal{B}_n(x_i,\varepsilon) \\ &\leq C^{q-1} \sum_i \mu(\mathcal{B}_n(x_i,\varepsilon))^q,\end{aligned}$$

where $C$ satisfies (5.30). Since $E$ is an arbitrary $(n,\varepsilon)$-generating set, for $q > 1$ we conclude that

$$\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu \leq C^{q-1}\Lambda(X,q,\varepsilon,n).$$

□

**Lemma 5.20.** *If $\mu$ satisfies the entropy doubling condition then for every sufficiently small $\varepsilon > 0$ and any $q < 1$ there exist constant $D_1$ such that for any $Z \subseteq X$ one has*

$$D_1\Lambda(Z,q,\varepsilon,n) \leq \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\,d\mu \leq \Lambda(X,q,\varepsilon/2,n)$$

*Proof.* We start with the second inequality. Let $E = \{x_i\}$ be an arbitrary $(n, \varepsilon/2)$-generating set. Then

$$\begin{aligned}\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu &\leq \sum_i \int_{\mathcal{B}_n(x_i,\varepsilon/2)} \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu \\ &\leq \sum_i \mu(\mathcal{B}_n(x_i,\varepsilon/2)^q,\end{aligned}$$

since $\mathcal{B}_n(x_i,\varepsilon/2) \subseteq \mathcal{B}_n(x,\varepsilon)$ for $x \in \mathcal{B}_n(x_i,\varepsilon/2)$. Thus

$$\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu \leq \Lambda(X,q,\varepsilon/2,n).$$

Let $E = \{x_i\}$ be an arbitrary maximal $(n, 2\varepsilon)$-separated set. Thus $\mathcal{B}_n(x_i,\varepsilon) \cap \mathcal{B}_n(x_j,\varepsilon) = \varnothing$ for $i \neq j$, and

$$\begin{aligned}\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu &\geq \sum_i \int\limits_{\mathcal{B}_n(x_i,\varepsilon)} \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu \\ &\geq \sum_i \mu(\mathcal{B}_n(x_i,2\varepsilon))^{q-1}\mu(B_n(x_i,\varepsilon)) \qquad (5.31)\\ &\geq \frac{1}{C}\sum_i \mu(\mathcal{B}_n(x_i,2\varepsilon))^q,\end{aligned}$$

where we have used the fact the for $x \in \mathcal{B}_n(x_i,\varepsilon)$ one has $\mathcal{B}_n(x,\varepsilon) \subseteq \mathcal{B}_n(x_i,2\varepsilon)$, and as in the proof of the previous lemma, $C > 1$ satisfies (5.30).

Let $Z \subseteq X$ and $\mathcal{G} = \{\mathcal{B}_n(z_j,\varepsilon)\}$ be an arbitrary centered covering of $Z$. Let $C^*$ be such that

$$\sup_x \frac{\mu(\mathcal{B}_n(x,4\varepsilon))}{\mu(\mathcal{B}_n(x,2\varepsilon))} < C^* \quad \text{and} \quad \sup_x \frac{\mu(\mathcal{B}_n(x,2\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon))} < C^*$$

for all $n \geq 1$. For any $j$ we fix some $i = i(j)$ such that such that $z_j \in \mathcal{B}_n(x_i,2\varepsilon)$. Denote the set of those $i$ by $I' = \{i(j) : j = 1,\ldots,\#(\mathcal{G})\}$. Firstly, consider $q \in [0,1)$. Since $\mathcal{B}_n(z_j,\varepsilon) \subseteq \mathcal{B}_n(x_i,4\varepsilon)$, we conclude that

$$\sum_j \mu(\mathcal{B}_n(z_j,\varepsilon))^q \leq \sum_i \mu(\mathcal{B}_n(x_i,4\varepsilon))^q \leq C^{*q}\sum_i \mu(\mathcal{B}_n(x_i,2\varepsilon))^q.$$

Secondly, consider $q < 0$. If $z_j \in \mathcal{B}_n(x_i,2\varepsilon)$, then $\mathcal{B}_n(x_i,2\varepsilon) \subseteq \mathcal{B}_n(z_j,4\varepsilon)$. Therefore,

$$\sum_i \mu(\mathcal{B}_n(x_i,2\varepsilon))^q \geq \sum_j \mu(\mathcal{B}_n(z_j,4\varepsilon))^q \geq C^{*2q}\sum_j \mu(\mathcal{B}_n(z_j,\varepsilon))^q.$$

In both cases, there exists some constant $D$ such that

$$\sum_i \mu(\mathcal{B}_n(x_i,2\varepsilon))^q \geq D\sum_j \mu(\mathcal{B}_n(z_j,\varepsilon))^q \geq D\ \Lambda(Z,q,\varepsilon,n).$$

Comparing this to (5.31) we conclude that there exists $D_1$ such that for any $Z \subseteq X$

$$\int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\ d\mu \geq D_1\ \Lambda(Z,q,\varepsilon,n).$$

This finishes the proof. $\square$

As a direct consequence of the above estimates we obtain the following result.

**Lemma 5.21.** *Suppose $\mu$ satisfies the entropy doubling condition, then for $q > 1$*

$$\overline{Ch}_\mu(f,q,X,2\varepsilon) \leq \limsup_{n\to\infty} \frac{1}{n} \log \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\ d\mu \leq \overline{Ch}_\mu(f,q,X,\varepsilon),$$

*and for $q < 1$*

$$\underline{Ch}_\mu(f,q,X,\varepsilon) \leq \liminf_{n\to\infty} \frac{1}{n} \log \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\ d\mu \leq \underline{Ch}_\mu(f,q,X,\varepsilon/2).$$

Therefore for $q < 1$ we have

$$H_\mu(f,q) = \lim_{\varepsilon\to0} \liminf_{n\to\infty} -\frac{1}{(q-1)n} \log \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\ d\mu = -\frac{1}{q-1}\underline{Ch}_\mu(f,q,X),$$

and for $q > 1$

$$H_\mu(f,q) = \lim_{\varepsilon\to0} \liminf_{n\to\infty} -\frac{1}{(q-1)n} \log \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\ d\mu = -\frac{1}{q-1}\overline{Ch}_\mu(f,q,X).$$

In both cases one has

$$(1-q)H_\mu(f,q) \geq h_\mu(f,q,X),$$

and hence we easily obtain a third version of our main result on the multifractal analysis of local entropies.

**Theorem 5.22.** *Let $f$ be a continuous transformation of a compact metric space $(X,d)$ with finite topological entropy. Consider an invariant non-atomic Borel probability measure $\mu$ which satisfies the entropy doubling condition. Then there exist $\underline{\alpha}, \overline{\alpha}$ such that*

*i)* $K_\alpha = \varnothing$ *for* $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$*;*

*ii) for* $\alpha \in (\underline{\alpha}, \overline{\alpha})$ *one has*

$$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) \leq \inf_q (q\alpha + T(q)) = T^*(\alpha),$$

*where* $T(q) = (1-q)H_\mu(f,q)$.

Once more, let us recall that Theorem 5.18 follows from Theorem 5.16 and the inequalities

$$h_\mu(f,q,X) \le \underline{Ch}_\mu(f,q,X) \le \overline{Ch}_\mu(f,q,X),$$

while for measures satisfying the entropy doubling condition, Theorem 5.22 follows from Theorem 5.16 (or Theorem 5.16) and the inequalities

$$h_\mu(f,q,X) \le \underline{Ch}_\mu(f,q,X) \le (1-q)H_\mu(q) \le \overline{Ch}_\mu(f,q,X).$$

At this moment we define those systems $(X,\mu,f)$ for which the multifractal formalism is valid.

**Definition 5.23.** *We say that a multifractal formalism for local entropies is valid for the dynamical system $(X,\mu,f)$ if there exist $\underline{\alpha},\overline{\alpha}$ such that:*

*i)* $K_\alpha = \varnothing$ *for* $\alpha \notin [\underline{\alpha},\overline{\alpha}]$ *and* $K_\alpha \neq \varnothing$ *for* $\alpha \in (\underline{\alpha},\overline{\alpha})$*;*

*ii)* $\mathcal{E}(\alpha) = h_{top}(f,K_\alpha)$ *is a smooth concave function of* $\alpha$ *on the interval* $(\underline{\alpha},\overline{\alpha})$*;*

*iii) let* $T(q) = (1-q)H_\mu(f,q)$*, then*

$$\mathcal{E}(\alpha) = \inf_q (q\alpha + T(q)) = T^*(\alpha)$$

*for all* $\alpha \in (\underline{\alpha},\overline{\alpha})$

## 5.11 Discussion of another entropy characteristic

The definition of the $(q,\mu)$-entropy involves coverings with the centers in a given set. This suits our technical purposes. One could argue however, how natural this definition is. In this section, we discuss a different characteristic, which is a particular case of the general Carathéodory dimension, see [17].

The notions and the results of this section are not necessary for understanding of the later results.

Again, let $Z \subseteq X$, $Z \neq \varnothing$, and $q,t \in \mathbb{R}$. We define

$$m_\mu(Z,q,t,\varepsilon,n) = \inf_{\mathcal{G}} \sum_i \mu(\mathcal{B}_{n_i}(x_i,\varepsilon))^q \exp(-tn_i),$$

where infimum is taken over all collections $\mathcal{G} = \{\mathcal{B}_{n_i}(x_i,\varepsilon)\}$ with $n_i \ge n$ covering $Z$, i.e., $Z \subseteq \cup_i \mathcal{B}_{n_i}(x_i,\varepsilon)$. Note that we do not require the centers $x_i$ to be in $Z$. Thus the quantities $m_\mu(Z,q,t,\varepsilon,n)$ are monotonic with respect to the first argument. Moreover,

$$m_\mu(Z,q,t,\varepsilon,n) \le M^c_\mu(Z,q,t,\varepsilon,n), \tag{5.32}$$

since we have only enlarged the class of admissible coverings and therefore the infimum cannot be larger.

Obviously, $m_\mu(Z,q,t,\varepsilon,n)$ is non-decreasing with $n$, therefore the following limit exists

$$m_\mu(Z,q,t,\varepsilon) = \lim_{n\to\infty} m_\mu(Z,q,t,\varepsilon,n) = \sup_{n\geq 1} m_\mu(Z,q,t,\varepsilon,n).$$

The qualitative behavior of $m_\mu(Z,q,t,\varepsilon)$ is similar to that of $M_\mu(Z,q,t,\varepsilon)$. Namely, there exists a critical value $h^*_\mu(f,q,Z,\varepsilon)$ such that

$$m_\mu(Z,q,t,\varepsilon) = \begin{cases} +\infty, & t < h^*_\mu(f,q,Z,\varepsilon), \\ 0, & t > h^*_\mu(f,q,Z,\varepsilon). \end{cases}$$

Taking into account (5.32) we get

$$m_\mu(Z,q,t,\varepsilon) \leq M^c_\mu(Z,q,t,\varepsilon) \leq \sup_{Z'\subseteq Z} M^c_\mu(Z',q,t,\varepsilon) = M_\mu(Z,q,t,\varepsilon).$$

Therefore $h^*_\mu(f,q,Z,\varepsilon) \leq h_\mu(f,q,Z,\varepsilon)$, and

$$h^*_\mu(f,q,Z) := \limsup_{\varepsilon\to 0} h^*_\mu(f,q,Z,\varepsilon) \leq h_\mu(f,q,Z).$$

The next lemma gives sufficient conditions for the equality of $h^*_\mu(f,q,Z)$ and $h_\mu(f,q,Z)$.

**Lemma 5.24.** *The equality $h^*_\mu(f,q,Z) = h_\mu(f,q,Z)$ holds for every $Z \subseteq X$, if $q \leq 0$, or if $q > 0$ and $\mu$ satisfies the entropy doubling condition.*

*Proof.* Suppose $h^*_\mu(f,q,Z) < h_\mu(f,q,Z)$. Since $h^*_\mu(f,q,Z) = \limsup\limits_{\varepsilon\to 0} h^*_\mu(f,q,Z,\varepsilon)$ and $h_\mu(f,q,Z) = \limsup_{\varepsilon\to 0} h_\mu(f,q,Z,\varepsilon)$, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$h^*_\mu(f,q,Z,\varepsilon/2) + \delta < h_\mu(f,q,Z,\varepsilon).$$

This implies that

$$m_\mu(Z,q,h^*_\mu(f,q,Z)+\delta,\varepsilon/2) = 0 \quad \text{and} \quad M_\mu(Z,q,h^*_\mu(f,q,Z)+\delta,\varepsilon) = \infty. \tag{5.33}$$

Denote for simplicity $h^*_\mu(f,q,Z) + \delta$ by $t$ and let $C$ be any number greater than 1. From (5.33) we conclude, that there exists a set $Z' \subseteq Z$ such that $M^c_\mu(Z',q,t,\varepsilon) > C^{2|q|}$. Hence there exists an integer $N$ such that

$$F_\mu(\mathcal{G}',q,t) = \sum_i \mu(\mathcal{B}_{n_i}(z_i,\varepsilon))^q \exp(-n_i t) > C^{2|q|} \tag{5.34}$$

for any collection $\mathcal{G}' = \{\mathcal{B}_{n_i}(z_i,\varepsilon)\}$ with $n_i \geq N$ and $z_i \in Z'$ covering $Z'$.

On the other hand, since $m_\mu(Z',q,t,\varepsilon/2) = 0$, there exists a collection $\mathcal{G}' = \{\mathcal{B}_{n_i}(x_i,\varepsilon/2)\}$ covering $Z$ with $n_i \geq N$ such that

$$F_\mu(\mathcal{G},q,t) = \sum_i \mu(\mathcal{B}_{n_i}(x_i,\varepsilon/2))^q \exp(-n_i t) < \frac{1}{2C}. \tag{5.35}$$

Without loss of generality we may assume that $\mathcal{B}_{n_i}(x_i, \varepsilon/2) \cap Z' \neq \varnothing$. Otherwise, after deleting these sets from $\mathcal{G}$ we obtain a smaller covering with this property, which still satisfies (5.35).

For any $i$ choose some $z_i \in \mathcal{B}_{n_i}(x_i, \varepsilon/2) \cap Z'$. Obviously, $\mathcal{B}_{n_i}(x_i, \varepsilon/2) \subseteq \mathcal{B}_{n_i}(z_i, \varepsilon)$. Therefore from (5.34), we obtain that

$$F_\mu(\mathcal{G}', q, t,) = \sum_i \mu(\mathcal{B}_{n_i}(z_i, \varepsilon))^q \exp(-tn_i) > C^{2|q|}.$$

Now we have to consider two cases: $q \leq 0$ and $q > 0$. If $q \leq 0$ then $\mu(\mathcal{B}_{n_i}(x_i, \varepsilon/2))^q \geq \mu(\mathcal{B}_{n_i}(z_i, \varepsilon))^q$ and hence

$$1 \leq C^{2|q|} < F_\mu(\mathcal{G}', q, t) \leq F_\mu(\mathcal{G}, q, t) < \frac{1}{2C} < \frac{1}{2}.$$

Thus we have arrived to a contradiction.

If $q > 0$ then the previous argument does not work. However, since $\mu$ satisfies the entropy doubling condition, we can choose $C > 1$ such that

$$\sup_x \frac{\mu(\mathcal{B}_n(x, \varepsilon))}{\mu(\mathcal{B}_n(x, \varepsilon/2))} < C$$

for all $n$. Therefore,

$$\mu(\mathcal{B}_{n_i}(z_i, \varepsilon)) \leq C\mu(\mathcal{B}_{n_i}(z_i, \varepsilon/2)) \leq C\mu(\mathcal{B}_{n_i}(x_i, \varepsilon)) \leq C^2\mu(\mathcal{B}_{n_i}(x_i, \varepsilon/2)),$$

since $\mathcal{B}_{n_i}(z_i, \varepsilon/2) \subseteq \mathcal{B}_{n_i}(x_i, \varepsilon)$ if $z_i \in \mathcal{B}_{n_i}(x_i, \varepsilon/2)$. The above inequality implies that for $q > 0$ one has

$$C^{2|q|} < F_\mu(\mathcal{G}', q, t') \leq C^{2q} F_\mu(\mathcal{G}, q, t) \leq \frac{C^{2q}}{2C},$$

and we have arrived to a contradiction again. $\square$

It is an important open question whether the result of the previous lemma, namely, the equality $h^*_\mu(f, q, Z) = h_\mu(f, q, Z)$ for $q > 0$, holds for measures which do not satisfy the entropy doubling condition.

## 5.12 Examples

In this section we discuss 3 examples which will illustrate the general results obtained in the previous section. We start with the following simple example.

### 5.12.1 Homogeneous measures.

**Definition 5.25 ([26]).** *Let $f$ be a continuous transformation of a compact metric space $(X, d)$. A Borel measure $\mu$ on $X$ is said to be $f$–homogeneous if for each $\varepsilon > 0$ there exist $\delta > 0$ and $c > 0$ such that*

$$\mu(\mathcal{B}_n(y, \delta)) \leq c\mu(\mathcal{B}_n(x, \varepsilon))$$

*for all $n \geq 1$ and $x, y \in X$.*

Maybe the simplest example of the homogeneous measure is Lebesgue measure invariant under the Arnold-Thom cat map of the torus $\mathbb{T}^2$, i.e., a linear automorphism of $\mathbb{T}^2$ given by a matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The same property is enjoyed by the Bowen-Margulis measure for any Anosov system.

This condition is rather strong. In particular, it implies that $\mu$ is a measure of maximal entropy and that

$$h_\mu(f,x) = h_\mu(f) = h_{top}(f)$$

for all $x \in X$. Thus the multifractal spectrum of local entropies for a homogeneous measure is trivial:

$$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) = \begin{cases} h_{top}(f), & \text{if } \alpha = h_{top}(f) \\ -\infty, & \text{otherwise.} \end{cases}$$

Not surprisingly, the spectrum of correlation entropies is quite simple as well

$$H_\mu(f,q) \equiv h_{top}(f).$$

Obviously, $\mathcal{E}(\alpha)$ and $(1-q)H_\mu(f,q)$ are related by the Legendre transform. However the interval $(\underline{\alpha}, \overline{\alpha})$ is empty. Thus, in this clearly degenerate situation, we can say that the multifractal formalism is valid for the homogeneous measures. According to some definitions in the physics literature, the homogeneous measures are not multifractal measures.

### 5.12.2 Gibbs measures for expansive homeomorphisms with specification

This is the most general situation known when the multifractal formalism for local entropies, according to the definition 5.23 is valid. Here we describe results obtained in [23].

First we introduce the notion of an expansive homeomorphisms with specification and Gibbs measures.

**Definition 5.26.** *A homeomorphism $f : X \to X$ is called expansive if there exists a constant $\gamma > 0$ such that if*

$$d(f^n(x), f^n(y)) < \gamma \ \textit{ for all } n \in \mathbb{Z} \quad \textit{then} \quad x = y.$$

*Such $\gamma$ is called an expansivity constant.*

Another important requirement is the following.

**Definition 5.27 ([5]).** *We say that a homeomorphism $f : X \to X$ has the specification property if for each $\delta > 0$ there exists an integer $p = p(\delta)$ such that the following holds:*
*if*

*a)* $I_1, \dots, I_n$ *are intervals of integers,* $I_j \subseteq [a,b]$ *for some* $a,b \in \mathbb{Z}$ *and all* $j$*,*

*b)* $dist(I_i, I_j) \geq p(\delta)$ *for* $i \neq j$*,*

*then for arbitrary* $x_1, \dots, x_n \in X$ *there exists a point* $x \in X$ *such that*

*1)* $f^{b-a+p(\delta)}(x) = x$*,*

*2)* $d(f^k(x), f^k(x_i)) < \delta$ *for* $k \in I_i$*.*

We denote these maps by 'homeomorphisms with specification'.

The specification property guarantees good mixing properties of $f$ and the existence of a sufficient number of periodic orbits.

We consider Gibbs measures for a special, but quite large, class of potentials $\mathcal{V}_f(X)$ which is defined as follows. We say that $\varphi \in \mathcal{V}_f(X)$ if it is continuous and there exist $\varepsilon > 0$ and $K > 0$ such that for all $n \in \mathbb{N}$

$$d(f^k(x), f^k(y)) < \varepsilon \text{ for } k = 0, \dots, n-1 \Rightarrow \left| \sum_{i=0}^{n-1} \varphi(f^i(x)) - \sum_{i=0}^{n-1} \varphi(f^i(y)) \right| < K.$$

For example, for a hyperbolic diffeomorphism $f$ any Hölder continuous function $\varphi$ is in $\mathcal{V}_f(X)$ [14, Prop.20.2.6].

The following theorem ensures existence and provides basic properties of the equilibrium states for the potentials from $\mathcal{V}_f(X)$. Basically it says that equilibrium states exist and are unique, moreover, they are Gibbs measures as well.

**Theorem 5.28 ([5, 11, 14]).** *If* $f$ *is an expansive homeomorphism with specification and* $\varphi \in \mathcal{V}_f(X)$ *then there exists a unique measure* $\mu_\varphi$*, called the equilibrium state of* $\varphi$*, such that*

$$P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi,$$

*where* $P(\varphi)$ *is the topological pressure of* $\varphi$*.*

*The equilibrium state* $\mu_\varphi$ *is a Gibbs measure as well, i.e., for sufficiently small* $\varepsilon > 0$ *there exist constants* $A_\varepsilon, B_\varepsilon > 0$ *such that for all* $x \in X$ *and* $n \geq 0$

$$A_\varepsilon \leq \frac{\mu_\varphi\left(\mathcal{B}_n(x,\varepsilon)\right)}{\exp\left(-nP(\varphi) + \sum_{i=0}^{n-1} \varphi(f^i(x))\right)} \leq B_\varepsilon. \tag{5.36}$$

*Moreover,* $\mu_\varphi$ *is ergodic, positive on open sets and mixing.*

The characteristic property of a Gibbs measure (5.36) is the key which allows us to perform a complete multifractal analysis of local entropies. It is easy to see from (5.36) that Gibbs measures satisfy the entropy doubling condition. Therefore according to the results in the previous sections it makes sense to look at the correlation entropies. The following result establishes the smoothness of the correlation entropies as a function of the parameter $q$.

**Lemma 5.29 ([23]).** *Suppose $f : X \to X$ is an expansive homeomorphism with specification, $\varphi \in \mathcal{V}_f(X)$ and $\mu$ is the corresponding Gibbs measure. Then*

$$(1-q)H_\mu(f,q) = P(q\varphi) - qP(\varphi).$$

*The function $(1-q)H_\mu(f,q)$ is convex and continuously differentiable; moreover, it is strictly convex provided $\mu$ is not a measure of maximal entropy.*

The results from the previous section say that the Legendre transform of $T(q) = (1-q)H_\mu(f,q)$ will give an upper estimate of the multifractal spectrum of local entropies. It turns out that there is an exact equality. The way to prove this, which is used in [23] (see also chapter 6), was adopted from [1, 2] and it consists of the following. It is quite clear that since the system $(f,\mu)$ is ergodic, there is only one value $\alpha_1$ such that $\mu(K_{\alpha_1}) = 1$, while for all other $\alpha$'s the sets $K_\alpha$ have measure 0. Actually, we know from the Brin-Katok formula that $\alpha_1 = h_\mu(f)$. Now since $h_{top}(f, K_\alpha) \leq \alpha$ for all $\alpha \geq 0$ and $h_{top}(f, K_\alpha) \geq h_\nu(f)$ for every invariant measure $\nu$ such that $\nu(K_\alpha) = 1$ (Theorem 5.3), taking $\alpha_1 = h_\mu(f)$ and $\nu = \mu$, we immediately conclude that

$$\alpha_1 \geq h_{top}(f, K_{\alpha_1}) \geq h_\mu(f) = \alpha_1.$$

Hence $h_{top}(f, K_{\alpha_1}) = \alpha_1$.

It turns out that we can generalize this argument to all other $\alpha$'s. The idea is to introduce a one-parameter family of measures $\mu_q$ which are Gibbs states for the potential $q\varphi$. Then for a suitable parameterization $\alpha(q)$

$$\mu_q(K_\alpha) = \begin{cases} 1, & \text{if } \alpha = \alpha(q), \\ 0, & \text{otherwise.} \end{cases}$$

This parameterization $\alpha(q)$ is built in the following way. Consider $T(q) = P(q\varphi) - qP(\varphi) = (1-q)H_\mu(f,q)$. $T(q)$ turns out to be a differentiable monotonically decreasing function of $q$. Let $\alpha(q) = -T'(q) \geq 0$ and $\mu_q$ be an equilibrium state for $\varphi_q = q\varphi$. Comparing (5.36) for two Gibbs measures $\mu$ and $\mu_q$ corresponding to the potentials $\varphi$ and $q\varphi$ we see that for every $x \in X$

$$h_{\mu_q}(f,x) = T(q) + qh_\mu(f,x)$$

provided at least one of the local entropies $h_\mu(f,x)$, $h_{\mu_q}(f,x)$ exist. Therefore if $x \in K_{\alpha(q)}$, i.e., $h_\mu(f,x) = \alpha(q)$, then

$$h_{\mu_q}(f,x) = T(q) + q\alpha(q).$$

It is not very difficult to verify that the measure-theoretic entropy of $\mu_q$ is given by

$$h_{\mu_q}(f) = T(q) + q\alpha(q).$$

Thus, since $h_{\mu_q}(f,x) = h_{\mu_q}(f)$ for $\mu_q$-a.e. $x$, and the equality holds only for $x \in K_{\alpha(q)}$, one gets that $\mu_q(K_{\alpha(q)}) = 1$. The final argument is similar to the one above for $\alpha = h_\mu(f)$. Namely, from Theorem 5.22 we know that for all $\alpha \geq 0$

$$h_{top}(f, K_\alpha) \leq T^*(\alpha) = \inf_q \big(T(q) + q\alpha\big).$$

On the other hand, since $\mu_q$ is an invariant measure concentrated on $K_{\alpha(q)}$, we obtain

$$h_{top}(f, K_{\alpha(q)}) \geq h_{\mu_q}(f) = T(q) + q\alpha(q).$$

Comparing the last two formulas we conclude that $h_{top}(f, K_{\alpha(q)}) = T^*(\alpha(q))$ for all $q$. To finish the proof we have to apply the result from [21], which states, that $\{\alpha(q)\}$ for $q \in \mathbb{R}$ exhausts the domain of the multifractal spectrum. More precisely, there exist an interval $[\underline{\alpha}, \overline{\alpha}]$ such that

$$K_\alpha = \varnothing \quad \text{ for } \alpha \notin [\underline{\alpha}, \overline{\alpha}],$$
$$\text{for every } \alpha \in (\underline{\alpha}, \overline{\alpha}) \text{ there exists } q \in \mathbb{R} \text{ such that } \alpha = \alpha(q).$$

This means that we obtained a complete description of a multifractal spectrum of local entropies for expansive homeomorphism with specification.

### 5.12.3 Weak Gibbs measures for interval maps with indifferent fixed points

For simplicity we consider a piecewise monotonic map with two full branches, expanding everywhere except one fixed point. The precise conditions are given in the next definition. This, and even more general situations, are considered in [13, 19, 28].

**Definition 5.30.** *We say that $f : [0,1] \to [0,1]$ is a Manneville-Pomeau type map if*

1) *$f$ is piecewise monotonic, and there exits $p > 0$ such that $f[0,p) = [0,1)$ and $f(p,1] = (0,1]$;*

2) *the branches $f|_{(0,p)}$ and $f|_{(p,1]}$ are $C^2$;*

3) *$f'(x) > 1$ for all $x > 0$ and $f'(x) \geq \lambda > 1$ for $x \in (p,1)$.*

4) *$f$ has the following asymptotic behavior when $x \to 0_+$*

$$f(x) = x + Cx^{1+\gamma}(1 + u(x))$$

*with constants $C > 0$, $\gamma \in (0,1)$, and where $u(x)$ a $C^1$ function such that $u'(x) = \mathcal{O}(x^{t-1})$ as $x \to 0_+$, for some $t > 0$.*

G.Pianigiani [18] was the first to show the existence of an absolutely continuous invariant measure for the Manneville-Pomeau type maps. We denote this measure by $\mu$. This invariant measure has positive entropy which is given by Rokhlin's formula

$$h_\mu(f) = \int \log |f'| d\mu.$$

The ergodic properties of $\mu$ have been studied extensively in the literature, e.g., [13, 15, 27, 12]. In particular, it has been recently established that for these systems correlations decay polynomially. Note that for the expanding systems the decay is exponential.

The absolutely continuous invariant measure $\mu$ is an equilibrium state for the potential $\varphi = -\log|f'|$. But it is not unique, the Dirac measure concentrated at 0 is an equilibrium state as well. The existence of two equilibrium states (i.e., non-uniqueness of equilibrium states) results in a singular behavior of the pressure function $P(q\varphi)$. Combining the results from [19, 24] we obtain the following statement.

**Lemma 5.31.** *Let $f$ be a Manneville-Pomeau type map, then $P(q\varphi)$ is a continuous, convex and non-increasing function. Moreover, $P(q\varphi) = 0$ for $q \geq 1$, $P(q\varphi) > 0$ for $q < 1$, $P(q\varphi)$ is a real-analytic function of $q$ for $q < 1$. At the critical point $q = 1$, one has the following asymptotics*

$$\frac{P(q\varphi)}{1-q} \to h_\mu(f) \ \text{as} \ q \nearrow 1.$$

The equilibrium state $\mu$ is not a Gibbs measure according to the standard definitions in ergodic theory (e.g., Theorem 5.28). The quotient

$$\frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\exp\left(\sum_{k=0}^{n-1}\varphi(f^k(x))\right)}$$

is not uniformly bounded from above and below by positive constants independent of $n$. However, for every sufficiently small $\varepsilon > 0$ one can find a sub-exponential sequence $K_n$, i.e., $\lim_{n\to\infty} \log K_n/n = 0$, such that

$$\frac{1}{K_n} \leq \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\exp\left(\sum_{k=0}^{n-1}\varphi(f^k(x))\right)} \leq K_n \tag{5.37}$$

for all $x$. This observation leads to the following definition.

**Definition 5.32 ([28]).** *We say that $\mu$ is a weak Gibbs measure if for every sufficiently small $\varepsilon > 0$ one can choose $K_n$ with $\lim_{n\to\infty} \log K_n/n = 0$ such that (5.37) holds for every $x$.*

The estimate (5.37) does not actually imply that $\mu$ satisfies the entropy doubling condition. Though it is clear that $\mu$ satisfies a weaker version of the doubling condition:

**Definition 5.33.** *We say that $\mu$ satisfies a weak entropy doubling condition if for every sufficiently small $\varepsilon > 0$ one has*

$$\lim_{n\to\infty} \frac{1}{n} \log \sup_x \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2))} = 0. \tag{5.38}$$

It is easy to check that after appropriate modifications of the proofs, all the result in section 5.10 remain valid. For example, from (5.38) we conclude that for every sufficiently small $\varepsilon > 0$ and any $\delta > 0$ there exist $N$ such that for all $n > N$ one has

$$C(\varepsilon, n) = \sup_x \frac{\mu(\mathcal{B}_n(x,\varepsilon))}{\mu(\mathcal{B}_n(x,\varepsilon/2))} \leq \exp{(\delta n)}.$$

Therefore the constants $C_1, C_2$ in the statement of Lemma 5.19 should be substituted by some numbers $C_1(n), C_2(n)$ which do not decay or grow too fast with $n$. In particular, there exists a constant $K > 0$, independent of $n$ and $\delta$, such that $C_1(n) > \exp(-K\delta n)$ and $C_2(n) < \exp(K\delta n)$ for sufficiently large $n$. As a result, for $q > 1$ we obtain that

$$\overline{Ch}_\mu(f,q,X,2\varepsilon) - K\delta \leq \limsup_{n\to\infty} \frac{1}{n} \log \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1}\, d\mu \leq \overline{Ch}_\mu(f,q,X,\varepsilon) + K\delta.$$

Since $\delta$ is an arbitrary positive number, we obtain exactly the same result as in Lemma 5.21.

As a result we conclude that the Legendre transform of $(1-q)H_\mu(f,q) = P(q\varphi)$ will give an upper estimate on the multifractal spectrum of local entropies.

The problem now is of course to show that we actually have equality between the multifractal spectrum and the Legendre transform of $(1-q)H_\mu(f,q)$. The method, used in the previous example, employs the construction of "reference measures" $\mu_q$. We have used the equilibrium states corresponding to the potentials $q\varphi$ as such measures. In this case we can repeat the same argument for $q \leq 1$ (corresponds to $\alpha \geq h_\mu(f)$) and achieve the same result. However for $q > 1$ (corresponds to $\alpha < h_\mu(f)$), the equilibrium states for $\varphi_q = q\varphi$ are just the Dirac measures, carrying zero entropy and are therefore useless for our purposes.

At the present moment it is not clear what is the structure of the multifractal spectrum for $\alpha < h_\mu(f)$. To answer this question, in our opinion, one should study in greater detail the pre-measures involved in the definition of topological entropy.

Finally, we would like to mention that in chapter 7 we will establish a relation between the notion of weak Gibbs measures as it appears in dynamical systems (definition 5.32), and the notion of weak Gibbs states accepted in statistical mechanics.

## 5.13 Concluding remarks

In the first part of this chapter we have constructed a general formalism which allows us to obtain an upper estimate on the multifractal spectrum of local entropies. Under additional assumptions (the entropy doubling condition) we have shown that the Legendre transform of the spectrum of correlation entropies gives an upper estimate. We have illustrated these results on 3 examples. For homogeneous measures the multifractal spectrum is a delta-like function. In the case of Gibbs measures for the expansive homeomorphisms with specification the multifractal spectrum is a smooth strictly concave function, which is equal to the Legendre transform of $(1-q)H_\mu(f,q)$. For the non-uniformly hyperbolic dynamical systems (interval maps with indifferent fixed points) the developed methods recover the multifractal spectrum only partially.

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# Chapter 6

# Multifractal Analysis of Local Entropies for Expansive Homeomorphisms with Specification

In the present chapter we study the multifractal spectrum of local entropies. We obtain results, similar to those of the multifractal analysis of pointwise dimensions, but under much weaker assumptions on the dynamical systems. We assume our dynamical system to be defined by an expansive homeomorphism with the specification property. We establish the variational relation between the multifractal spectrum and other thermodynamic characteristics of the dynamical system, including the spectrum of correlation entropies.

## 6.1 Introduction

Recently in the series of papers [12, 13, 1] L. Barreira, Ya.B. Pesin, J. Schmeling and H. Weiss performed a complete multifractal analysis of local dimensions, entropies and Lyapunov exponents for conformal expanding maps and surface Axion A diffeomorphisms for Gibbs measures. The main goal of these papers was primarily the analysis of the local (pointwise) dimensions. This is an extremely difficult problem and, for example, similar results for hyperbolic systems in dimensions 3 and higher have not yet been obtained.

In the present work we concentrate our attention on the multifractal analysis of the local (pointwise) entropies. We are able to obtain results, which are similar to those mentioned above, for Gibbs measures of the expansive homeomorphisms

This chapter is based on F. Takens, E. Verbitski, *Multifractal analysis of local entropies for expansive homeomorphisms with specification*, Commun. Math. Phys. **203**, 1999, pp.593-612.

with specification property. These dynamical system have strong chaotic properties. They may even not have Markov partitions which is a crucial condition in the previous works. However due to the fact that less is known about thermodynamical properties of these dynamical systems we were able to obtain only the continuous differentiability of the multifractal spectrum of local entropies (compare: the same spectrum for the dynamical systems with Markov partitions are analytic). We believe that the smoothness of the multifractal spectrum in our case can be improved.

We have related the multifractal spectrum of the local entropies to the the spectrum of correlation entropies. These correlation entropies serve as entropy-like analogs of the Hentschel–Procaccia and Renyi spectra of generalized dimensions. This allows us to complete the analogy between the multifractal analysis of local dimensions and entropies.

## 6.2 Expansiveness and Specification

The following definitions and fundamental results are taken from [5, 7, 18], for a compact presentation see [9, Chap.20].

Throughout this chapter we assume $(X, d)$ to be a compact metric space.

**Definition 6.1.** *A homeomorphism $f : X \to X$ is called expansive if there exists a constant $\gamma > 0$ such that if*

$$d(f^n(x), f^n(y)) < \gamma \text{ for all } n \in \mathbb{Z} \quad \text{then} \quad x = y. \tag{6.1}$$

*The maximal $\gamma$ with such property is called the expansivity constant.*

Another important property is the following.

**Definition 6.2 (R.Bowen [5]).** *We say that a homeomorphism $f : X \to X$ has the specification property (abbreviated to "with specification") if for each $\delta > 0$ there exists an integer $p = p(\delta)$ such that the following holds:*
*if*

*a) $I_1, \dots, I_n$ are intervals of integers, $I_j \subseteq [a, b]$ for some $a, b \in \mathbb{Z}$ and all $j$,*

*b) dist$(I_i, I_j) \geq p(\delta)$ for $i \neq j$,*

*then for arbitrary $x_1, \dots, x_n \in X$ there exists a point $x \in X$ such that*

*1) $f^{b-a+p(\delta)}(x) = x$,*

*2) $d(f^k(x), f^k(x_i)) < \delta$ for $k \in I_i$.*

The specification property guarantees good mixing and the existence of a sufficient number of periodic orbits. Homeomorphisms that are expansive and satisfy specification, form a general class of "strongly chaotic" dynamical systems. For example, the following holds:

**Theorem 6.3 (Theorem 18.3.9, [9]).** *Let $\Lambda$ be a topologically mixing compact locally maximal hyperbolic set for a diffeomorphism $f$. Then $f|_\Lambda$ has the specification property.*

## 6.3 Equilibrium states

For the multifractal analysis one needs an invariant probability measure. On an attractor there is usually one physically relevant measure (density of a generic orbit) called the SRB (Sinai-Ruelle-Bowen) measure which often belongs to the class of equilibrium states or Gibbs measures. We introduce these notions now. Again, let $(X, d)$ be a compact space, $f : X \to X$ a continuous map and $\varphi : X \to \mathbb{R}$ a continuous function. We shall use the following notation.

**Definition 6.4.** *For every $n \in \mathbb{N}$ and any $x, y \in X$ define a new metric*

$$d_n(x, y) = \max_{i=0,\dots,n-1} d(f^i(x), f^i(y)),$$

*and let $\mathcal{B}_n(x, \varepsilon) = \{y \in X : \ d_n(x, y) < \varepsilon\}$ for $\varepsilon > 0$.*

*The set $E \subseteq X$ is said to be $(n, \varepsilon)$-separated if for every $x, y \in E$ such that $x \neq y$ we have $d_n(x, y) > \varepsilon$.*

*We say that a set $F \subseteq X$ is $(n, \varepsilon)$-spanning if for every $y \in X$ there exist $x \in F$ such that $d_n(x, y) < \varepsilon$.*

*For any function $\varphi : X \to \mathbb{R}$ and $x \in X$ put*

$$(S_n\varphi)(x) = \sum_{k=0}^{n-1} \varphi(f^k(x)).$$

Now we introduce the topological pressure which will be defined on the space $C(X)$ of all continuous functions on $(X, d)$.

**Definition 6.5.** *For $n \in \mathbb{N}$ and $\varepsilon > 0$ define*

$$Z_n(\varphi, \varepsilon) = \sup_E \left\{ \sum_{x \in E} \exp\Big((S_n\varphi)(x)\Big) \right\}, \tag{6.2}$$

*where the supremum is taken over all $(n, \varepsilon)$-separated sets $E$. The pressure is then defined as*

$$P(\varphi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \varepsilon). \tag{6.3}$$

The topological entropy of $f$, denoted by $h_{top}(f)$, is by definition the topological pressure of $\varphi \equiv 0$. The topological pressure admits other equivalent definitions, for this see [22]. In particular, the following statement is known as the Variational Principle.

**Theorem 6.6.** *Denote by $\mathcal{M}_f(X)$ the set of all $f$-invariant Borel probability measures on $X$. Let $\varphi \in C(X)$. Then*

$$P(\varphi) = \sup_{\mu \in \mathcal{M}_f(X)} \left( h_\mu(f) + \int \varphi d\mu \right).$$

This result inspires the following definition.

**Definition 6.7.** *An element $\mu$ of $\mathcal{M}_f(X)$ is called an equilibrium state for the potential $\varphi$ if*

$$P(\varphi) = h_\mu(f) + \int \varphi \, d\mu.$$

An equilibrium state for $\varphi \equiv 0$ (if it exists) is called a measure of maximal entropy. We recall some other basic properties of the topological pressure:

1. $P : C(X) \to \mathbb{R}$ is continuous and monotonicly increasing, i.e.,

$$\varphi \leq \psi \Rightarrow P(\varphi) \leq P(\psi).$$

2. Only one of the following holds

$$\begin{aligned} P(\varphi) &= +\infty \quad \forall \varphi \in C(X), \\ P(\varphi) &< +\infty \quad \forall \varphi \in C(X). \end{aligned}$$

   Therefore to exclude the infinite case, it is sufficient to require that the topological entropy is finite. On the other hand, expansive homeomorphisms always have finite topological entropy.

3. $P : C(X) \to \mathbb{R}$ is convex, i.e., $\forall \lambda \in [0,1]$

$$P(\lambda\varphi + (1-\lambda)\psi) \leq \lambda P(\varphi) + (1-\lambda)P(\psi).$$

4. For any $\varphi \in C(X)$ and $c \in \mathbb{R}$ one has $P(\varphi + c) = P(\varphi) + c$.

Now we impose additional conditions on the class of potentials under consideration. We say that $\varphi \in \mathcal{V}_f(X)$ if it is continuous and there exist $\varepsilon > 0$ and $K > 0$ such that for all $n \in \mathbb{N}$

$$d(f^k(x), f^k(y)) < \varepsilon \text{ for } k = 0, \ldots, n-1 \Rightarrow \left|(S_n\varphi)(x) - (S_n\varphi)(y)\right| < K.$$

For example, for a hyperbolic diffeomorphism $f$ any Hölder continuous function $\varphi$ is in $\mathcal{V}_f(X)$ [9, Prop. 20.2.6].

**Theorem 6.8 ([5, 17, 9]).** *If $f$ is an expansive homeomorphism with specification and $\varphi \in \mathcal{V}_f(X)$ then there exists a unique measure $\mu = \mu_\varphi$ such that*

$$P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi.$$

*Moreover, $\mu_\varphi$ is ergodic, positive on open sets and mixing.*

The equilibrium state $\mu_\varphi$ can be constructed from the measures concentrated on periodic points in the following way. For every $n \geq 1$ define a probability measure

$\mu_{\varphi,n}$ supported on the set of periodic points $Fix(f^n) = \{x \in X : f^n(x) = x\}$ as follows

$$\mu_{\varphi,n} = \frac{1}{P(f,\varphi,n)} \sum_{x \in Fix(f^n)} e^{(S_n\varphi)(x)} \delta_x, \tag{6.4}$$

where $\delta_x$ is a unit measure at $x$ and $P(f,\varphi,n) = \sum\limits_{x \in Fix(f^n)} e^{(S_n\varphi)(x)}$ is a normalizing constant.

**Theorem 6.9 ([5, 9]).** *For $\varphi \in \mathcal{V}_f(X)$ the corresponding equilibrium state $\mu_\varphi$ is a vague limit of the sequence $\{\mu_{\varphi,n}\}_{n=1}^\infty$, i.e., for every $h \in C(X)$*

$$\int h(x) d\mu_{\varphi,n} \to \int h(x) d\mu_\varphi \quad \textit{as} \quad n \to \infty.$$

For the analysis of local entropies the following result will play a key role.

**Theorem 6.10 (Proposition 2.1 [7], Theorem 20.3.4 [9]).** *Let $f$ be an expansive homeomorphism with the specification property. Consider $\varphi \in \mathcal{V}_f(X)$ and its equilibrium state $\mu_\varphi$. Then for sufficiently small $\varepsilon > 0$ there exist constants $A_\varepsilon, B_\varepsilon > 0$ such that for all $x \in X$ and $n \geq 0$*

$$A_\varepsilon \leq \frac{\mu_\varphi\left(\{y \in X : d(f^k(x), f^k(y)) < \varepsilon \textit{ for } k = 0, \ldots, n-1\}\right)}{\exp\left(-nP(\varphi) + (S_n\varphi)(x)\right)} \leq B_\varepsilon. \tag{6.5}$$

*Remark.* Actually, the result above states that, for expansive homeomorphisms with specification, the equilibrium states are so-called Gibbs measures (states) as well. See [7] for detailed discussion.

We have seen that for every $\varphi \in \mathcal{V}_f(X)$ there exists a unique equilibrium state. Using (6.4) and (6.5) we are able to give necessary and sufficient conditions for potentials $\varphi, \psi \in \mathcal{V}_f(X)$ to have the same equilibrium states $\mu_\varphi = \mu_\psi$.

**Theorem 6.11.** *Let $f$ be an expansive homeomorphism with specification. The equilibrium states $\mu_\varphi$ and $\mu_\psi$ corresponding to the potentials $\varphi, \psi \in \mathcal{V}_f(X)$ coincide if and only if there exists a constant $c \in \mathbb{R}$ such that*

$$(S_n\varphi)(x) = (S_n\psi)(x) + nc \tag{6.6}$$

*for all $x \in Fix(f^n)$ and all $n$.*

*Proof.* If (6.6) holds for all $x \in Fix(f^n)$ and $n$ then by (6.4) one has $\mu_{\varphi,n} = \mu_{\psi,n}$ for all $n$. Thus $\mu_\varphi = \mu_\psi$.

Suppose that $\mu_\varphi = \mu_\psi =: \mu$. Consider "adjusted" potentials $\widetilde{\varphi} = \varphi - P(\varphi)$ and $\widetilde{\psi} = \psi - P(\psi)$. Let $x \in Fix(f^n)$ for some $n \in \mathbb{N}$, applying (6.5) for sufficiently small $\varepsilon > 0$ we conclude that

$$A_\varepsilon^\varphi \exp\big( (S_n\widetilde{\varphi})(x) \big) \leq \mu(\mathcal{B}_n(x,\varepsilon)) \leq B_\varepsilon^\psi \exp\big( (S_n\widetilde{\psi})(x) \big).$$

This implies that $(S_n\widetilde{\varphi})(x) \le (S_n\widetilde{\psi})(x) + C'$ for some constant $C'$ independent of $x$ and $n$. Since $x \in Fix(f^{kn})$ for all $k \in \mathbb{N}$, we have that

$$(S_n\widetilde{\varphi})(x) = \lim_{k\to\infty} \frac{(S_{kn}\widetilde{\varphi})(x)}{k} \le \lim_{k\to\infty} \frac{(S_{kn}\widetilde{\psi})(x)}{k} = (S_n\widetilde{\psi})(x).$$

By symmetry we obtain the opposite inequality. Hence

$$(S_n\widetilde{\varphi})(x) = (S_n\widetilde{\psi})(x)$$

for all $x \in Fix(f^n)$ and $n \in \mathbb{N}$. This implies (6.6) with $c = P(\varphi) - P(\psi)$. □

## 6.4 Thermodynamic formalism for expansive homeomorphisms with specification

In this section we establish some technical results on properties of equilibrium states for expansive homeomorphisms which will be exploited later in the proof of the main result.

**Lemma 6.12.** *Suppose $f : X \to X$ is an expansive homeomorphism with specification. Let $\varphi \in \mathcal{V}_f(X)$. Then the function $P(q\varphi)$ is continuously differentiable with respect to $q$ and its derivative is given by*

$$\frac{dP(q\varphi)}{dq} = \int \varphi d\mu_q,$$

*where $\mu_q$ is the equilibrium state corresponding to the potential $q\varphi$. Moreover, $P(q\varphi)$ is a strictly convex function of $q$ provided the equilibrium state $\mu_\varphi$ for $\varphi$ is not a measure of maximal entropy. In the case, $\mu_\varphi$ is the measure of maximal entropy one has $P(q\varphi) = qP(\varphi)(1-q)h_{top}(f)$ for all $q \in \mathbb{R}$.*

*Proof.* We shall use several results from [22] to show that $P(q\varphi)$ is a differentiable function of $q$. Our proof of this fact is similar to one which can be found in [10].

Assume for a moment that $f : X \to X$ is just a continuous map on a compact metric space $(X, d)$ with finite topological entropy. It is known that the topological pressure is a continuous and convex function on $C(X)$. Therefore, for every $\varphi, \psi \in C(X)$, the function

$$t \to \frac{P(\varphi + t\psi) - P(\varphi)}{t}$$

is decreasing as $t \downarrow 0$. Hence there exist right and left derivatives of $P(\varphi)$ in the direction of $\psi$, i.e.,

$$d^+P(\varphi)(\psi) = \lim_{t\to 0+} \frac{P(\varphi + t\psi) - P(\varphi)}{t},$$

$$d^-P(\varphi)(\psi) = \lim_{t\to 0-} \frac{P(\varphi + t\psi) - P(\varphi)}{t}.$$

We say that the pressure $P$ is Gâteaux differentiable at $\varphi$ if for every $\psi$ the following holds

$$d^+P(\varphi)(\psi) = d^-P(\varphi)(\psi).$$

A tangent functional (subdifferential) to $P(\cdot)$ at $\varphi$ is a linear functional $\alpha$ on $C(X)$ such that

$$P(\varphi+\psi) - P(\varphi) \geq \alpha(\psi)$$

for all $\psi \in C(X)$. Applying the Riesz representation theorem we conclude that there exist a finite signed measure $\nu$ on $X$ such that

$$\alpha(\psi) = \int \psi d\nu$$

for all $\psi \in C(X)$. From now on we identify the tangent functional $\alpha$ with the measure $\nu$ from the previous representation.

One can easily check that the pressure $P$ is Gâteaux differentiable at $\varphi$ if and only if there is a unique tangent functional $\nu$ to $P$ at $\varphi$ [22, Corrolary 2] and that

$$dP(\varphi)(\psi) = \int \psi d\nu.$$

Denote by $t_\varphi(P)$ the set of all tangent functionals to $P$ at $\varphi$ and by $\mathcal{M}_\varphi(X)$ the set of all equilibrium states corresponding to the potential $\varphi$. Applying the Variational Principle, one concludes

$$\mathcal{M}_\varphi(X) \subseteq t_\varphi(P).$$

Combining the results of Theorems 8.2 and 9.15 from [22], one has that for expansive homeomorphism $f : X \to X$

$$t_\varphi(X) = \mathcal{M}_\varphi(X)$$

for every $\varphi \in C(X)$.

Since for every $\varphi \in \mathcal{V}_f(X)$ the set $\mathcal{M}_\varphi(X)$ consists of a single element (uniqueness of equilibrium states), we have that the pressure $P$ is Gâteaux differentiable at any $\varphi \in \mathcal{V}_f(X)$ and

$$\frac{d}{dt}P(\varphi + t\psi)\Big|_{t=0} = \int \psi d\mu_\varphi \tag{6.7}$$

for all $\psi \in C(X)$. This proves the differentiability of the pressure at $q = 1$. The result for all other $q$ follows in the same manner since $q\varphi \in \mathcal{V}_f(X)$ for every $q \in \mathbb{R}$ if $\varphi \in \mathcal{V}_f(X)$.

To show the continuity of the derivative we have to establish that

$$\int \varphi d\mu_{q_n} \to \int \varphi d\mu_q \text{ as } n \to \infty$$

for every sequence $\{q_n\}$ with $q_n \to q$. We shall obtain this as a consequence of the fact that actually $\mu_{q_n} \to \mu_q$ in a weak$^*$–topology, i.e., $\int hd\mu_{q_n} \to \int hd\mu_q$ for every $h \in C(X)$.

Corollary 4 from [23] states that the pressure $P(\cdot)$ is Gâteaux differentiable at $\psi$ if and only if for an arbitrary sequence of invariant measures $\{\nu_n\}$ such that

$$h_{\nu_n}(f) + \int \psi d\nu_n \to P(\psi)$$

one has $\nu_n \to \nu$ in the weak$^*$–topology, where $\nu$ is the equilibrium state for $\psi$.

Consider an arbitrary sequence $\{q_n\} \in \mathbb{R}$ with $q_n \to q$ as $n \to \infty$. Let $\mu_{q_n}$ be the equilibrium state corresponding to $q_n\varphi$. Since the pressure is continuous on $C(X)$, we have that

$$P(q_n\varphi) = h_{\mu_{q_n}} + \int q_n\varphi d\mu_{q_n} \to P(q\varphi) \text{ as } n \to \infty.$$

But then

$$h_{\mu_{q_n}} + \int q\varphi d\mu_{q_n} = P(q_n\varphi) + (q - q_n)\int \varphi d\mu_{q_n} \to P(q\varphi) \text{ as } n \to \infty$$

as well, due to $|(q-q_n)\int \varphi d\mu_{q_n}| \le |q-q_n| ||\varphi||_{C^0} \to 0$. Since we have already proved that $P$ is Gâteaux differentiable at $q\varphi$, we conclude that $\mu_{q_n} \to \mu_q$ in the weak$^*$–topology and the statement follows.

Suppose, $\mu_\varphi$ is not a measure of maximal entropy. Then applying the result of Theorem 6.11 we conclude that the equilibrium states $\mu_{q_1}$ and $\mu_{q_2}$, corresponding to the potentials $q_1\varphi$ and $q_2\varphi$ respectively, are not equal if $q_1 \ne q_2$. Indeed, assume $\mu_{q_1} = \mu_{q_2}$ for some $q_1 \ne q_2$. Then by Theorem 6.11 we conclude that for some constant $c$

$$(S_n q_1\varphi)(x) = (S_n q_2\varphi)(x) + nc$$

for all $n$ and $x \in Fix(f^n)$. This implies that $(S_n\varphi)(x) = n\tilde{c}$ with $\tilde{c} = c/(q_1 - q_2)$. Applying again Theorem 6.11, one has that the equilibrium state $\mu_\varphi$ and the equilibrium state $\mu_0$, corresponding to the potential $\psi \equiv 0$, are equal. This means that $\mu_\varphi$ is the measure of maximal entropy. Hence we have arrived to a contradiction with the assumption. Therefore $\mu_{q_1} \ne \mu_{q_2}$ if $q_1 \ne q_2$.

Note that if a function $h : \mathbb{R} \to \mathbb{R}$ is strictly convex, then for every $q_0 \in \mathbb{R}$ there exists $\lambda(q_0) \in \mathbb{R}$ such that

$$h(q) > h(q_0) + \lambda(q_0)(q - q_0) \quad \text{ for all } q \ne q_0.$$

If we take $\lambda(q_0) = \int \varphi d\mu_{q_0}$, $q_0 \in \mathbb{R}$, then since $\mu_q \ne \mu_{q_0}$ and $\mu_q$ is the unique equilibrium state for $q\varphi$, one has

$$\begin{aligned}
P(q\varphi) &= h_{\mu_q}(f) + \int q\varphi d\mu_q \\
&= \sup_{\mu \in \mathcal{M}_f(X)} \Big(h_\mu(f) + \int q\varphi d\mu\Big) \\
&> h_{\mu_{q_0}}(f) + \int q\varphi d\mu_{q_0} \\
&= h_{\mu_{q_0}}(f) + \int q_0\varphi d\mu_{q_0} + (q - q_0)\int \varphi d\mu_{q_0} \\
&= P(q_0\varphi) + \lambda(q_0)(q - q_0).
\end{aligned}$$

This means that $P(q\varphi)$ is a strictly convex function.

If the equilibrium state $\mu_\varphi$ is indeed a measure of maximal entropy, then $\mu_\varphi = \mu_{q\varphi} =: \mu$ for all $q \in \mathbb{R}$. This is a consequence of Theorems 6.11 and 6.8. Then applying the Variational Principle to $\mu_\varphi$ and $\mu_{q\varphi}$ we conclude that

$$P(q\varphi) = h_\mu(f) + q\int \varphi d\mu,$$
$$P(\varphi) = h_\mu(f) + \int \varphi d\mu,$$

where $h_\mu(f) = h_{top}(f)$ since $\mu$ is the measure of maximal entropy. And the result follows immediately. □

*Remark.* Much stronger result on smoothness of the pressure are known. For example, the analyticity of pressure has been established for Smale spaces [17], i.e., generalizations of Axiom A diffeomorphisms. The key property, which these systems inherit from hyperbolic dynamical systems, is the so-called local product structure which in its turn guarantees the existence of Markov partitions. The known methods of establishing the analyticity of pressure strongly rely on this Markov structure. Expansive homeomorphism with specification do not necessarily have Markov partitions. We were able to prove only the differentiability of the pressure for expansive homeomorphism. However we believe that this result can be improved.

We say that $E$ is a maximal $(n, \varepsilon)$-separated set if it can not be enlarged by adding new points preserving the separation condition. It is easy to see that every maximal $(n, \varepsilon)$-separated set $E$ is an $(n, \varepsilon)$-spanning set as well.

The following estimates from [7] will be used later.

**Lemma 6.13.** *Let $f$ be an expansive homeomorphisms and $\gamma > 0$ be its expansivity constant. Let $\varphi \in \mathcal{V}_f(X)$. For every finite set $E$ put*

$$Z_n(\varphi, E) = \sum_{x\in E} \exp\Big((S_n\varphi)(x)\Big).$$

1) *If $\varepsilon, \varepsilon' < \gamma/2$ and $E, E'$ are the maximal $(n, \varepsilon)$- and $(n, \varepsilon')$-separated sets respectively then one has*

   $$Z_n(\varphi, E) \le C Z_n(\varphi, E'),$$

   *where the constant $C = C(\varepsilon, \varepsilon')$ is independent of $n$. In particular,*

   $$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, E_n), \tag{6.8}$$

   *where $E_n$ are maximal $(n, \varepsilon)$-separated sets.*

2) *If, furthermore, $f$ satisfies the specification property and $\varepsilon < \gamma/2$ then there exists a constant $D = D(\varphi, \varepsilon) > 0$ such that*

   $$|\log Z_n(\varphi, E_n) - nP(\varphi)| < D \tag{6.9}$$

   *for every $n$ and each maximal $(n, \varepsilon)$-separated set $E_n$.*

## 6.5 Topological entropy for non-compact sets

The generalization of the topological entropy to non-compact or non-invariant sets goes back to Bowen [4]. Later Pesin and Pitskel [15] generalized the notion of pressure to the case of non-compact sets. Note that by definition topological entropy is the topological pressure for $\varphi \equiv 0$. Now we give the formal definition of the topological entropy of a non-compact or non-invariant set.

Suppose $f : X \to X$ is a continuous map on a compact metric space $(X, d)$. Let $\mathfrak{U} = \{U_1, \dots, U_M\}$ be a finite open cover of $X$. By definition, a string $\mathbf{U}$ is a sequence $U_{i_1} \dots U_{i_n}$ with $i_k \in \{1, \dots, M\}$, its length $n$ is denoted by $n(\mathbf{U})$. The collection of all strings of length $n$ is denoted by $W_n(\mathfrak{U})$. For each $\mathbf{U} \in W_n(\mathfrak{U})$ define the open set

$$\begin{aligned} X(\mathbf{U}) &= U_1 \cap f^{-1}U_2 \cap \dots \cap f^{-n+1}U_n \\ &= \{x \in X : \ f^{k-1}x \in U_k, \ k = 1, \dots, n\}. \end{aligned}$$

We say that a collection of stings $\Gamma$ covers a set $Z \subseteq X$ if

$$\bigcup_{\mathbf{U} \in \Gamma} X(\mathbf{U}) \supset Z.$$

For every real number $s$ introduce

$$M(Z, s, \mathfrak{U}) = \lim_{n \to \infty} \inf_{\Gamma} \sum_{\mathbf{U} \in \Gamma} \exp(-n(\mathbf{U})s),$$

where the infimum is taken over all collections $\Gamma \in \bigcup_{k \geq n} W_k(\mathfrak{U})$ covering $Z$. There exists a unique value $s$ such that $M(Z, \cdot, \mathfrak{U})$ jumps from $+\infty$ to 0

$$h(Z, \mathfrak{U}) := s = \sup\{s : M(Z, s, \mathfrak{U}) = +\infty\} = \inf\{s : M(Z, s, \mathfrak{U}) = 0\}.$$

Finally, one can show that the following limit exists

$$h_{top}(f|_Z) := \lim_{\operatorname{diam}(\mathfrak{U}) \to 0} h(Z, \mathfrak{U}).$$

**Definition 6.14.** *The number $h_{top}(f|_Z)$ is called the topological entropy of $f$ restricted to the set $Z$, or simply the topological entropy of $Z$.*

This definition of the topological entropy is similar to the definition of the Hausdorff dimension (the diameters of the covering open sets are substituted by $\exp(-n(\mathbf{U}))$ which can be considered as a "dynamical diameter" of $\mathbf{U}$). Indeed, these definitions are particular cases of the so-called Carathéodory dimension characteristics [11].

**Theorem 6.15.** *([14]) The topological entropy as defined above has the following properties:*

*1) $h_{top}(f|_{Z_1}) \leq h_{top}(f|_{Z_2})$ for any $Z_1 \subseteq Z_2 \subseteq X$;*

*2) $h_{top}(f|_Z) = \sup_i h_{top}(f|_{Z_i})$ where $Z = \cup_{i=1}^{\infty} Z_i \subseteq X$;*

*3) if $\mu$ is an invariant measure such that $\mu(Z) = 1$ then $h_{top}(f|_Z) \geq h_\mu(f)$.*

## 6.6 Local entropies

In this section we give the definition of local entropy. The fundamental result on its existence almost everywhere is the Brin–Katok formula below.

Using the notation from section 3 we introduce the lower and upper local entropies at $x \in X$ as follows

$$\underline{h}_\mu(f,x) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x,\varepsilon)), \tag{6.10}$$

$$\overline{h}_\mu(f,x) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(\mathcal{B}_n(x,\varepsilon)). \tag{6.11}$$

Note that the limits in $\varepsilon$ exist due to the monotonicity.

We say that the local entropy exists at $x$ if

$$\underline{h}_\mu(f,x) = \overline{h}_\mu(f,x). \tag{6.12}$$

In this case the common value will be denoted by $h_\mu(f,x)$.

**Theorem 6.16 (Brin–Katok formula, [6]).** *Let $f : X \to X$ be a continuous map on a compact metric space $(X,d)$ preserving a non-atomic Borel measure $\mu$. Then*

*1) for $\mu$-a.e. $x \in X$ the local entropy exists, i.e.,*

$$h_\mu(f,x) = \underline{h}_\mu(f,x) = \overline{h}_\mu(f,x);$$

*2) $h_\mu(f,x)$ is an $f$–invariant function of $x$ and*

$$\int h_\mu(f,x)\, d\mu = h_\mu(f),$$

*where $h_\mu(f)$ is the measure–theoretic entropy of $f$.*

*Remark.* If $\mu$ is ergodic then $h_\mu(f,x) = h_\mu(f)$ for $\mu$-a.e. $x \in X$.

**Lemma 6.17.** *Let $f$ be an expansive homeomorphism with specification. Consider an equilibrium state $\mu_\varphi$ for the potential $\varphi \in \mathcal{V}_f(X)$. For every $x \in X$ put*

$$\underline{\varphi}^*(x) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)),$$

$$\overline{\varphi}^*(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)).$$

*Then*

$$\underline{h}_\mu(f,x) = P(\varphi) - \overline{\varphi}^*(x),$$
$$\overline{h}_\mu(f,x) = P(\varphi) - \underline{\varphi}^*(x),$$

*for all $x \in X$. Therefore*

$$\underline{h}_\mu(f,x) = \overline{h}_\mu(f,x) \quad \textit{if and only if} \quad \underline{\varphi}^*(x) = \overline{\varphi}^*(x).$$

*Proof.* Using the estimate from Theorem 6.10 we conclude that for every sufficiently small $\varepsilon > 0$ and some constants $\underline{c}, \overline{c}$ one has

$$\frac{\underline{c}}{n} + P(\varphi) - \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) \leq -\frac{1}{n}\log\mu(\mathcal{B}_n(x,\varepsilon)) \leq \frac{\overline{c}}{n} + P(\varphi) - \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))$$

for all $n \geq 1$ and every $x \in X$. The statement follows easily. $\square$

## 6.7 Multifractal spectrum for local entropies

Following [1] we introduce a multifractal spectrum for (local) entropies. For every $\alpha$ consider the level set of local entropy

$$K_\alpha = \{x \in X : \ h_\mu(f,x) = \alpha\}. \tag{6.13}$$

and the corresponding multifractal decomposition in level sets

$$X = \bigcup_\alpha K_\alpha \bigcup \{x \in X : \ h_\mu(f,x) \text{ does not exist}\}. \tag{6.14}$$

We use the topological entropy, defined in section 6.5, to measure the "size" of sets $\{K_\alpha\}$. Namely, define the multifractal spectrum for local entropies as follows

$$\mathcal{E}_E(\alpha) = h_{top}(f|_{K_\alpha}). \tag{6.15}$$

This notation needs a brief explanation: the two E's stand for the topological $\mathcal{E}$ntropy of level set of local *E*ntropy. For other multifractal spectra $\mathcal{D}_E, \mathcal{E}_D, \mathcal{D}_D$ see [1].

From a general multifractal formalism one expects $\mathcal{E}_E(\alpha)$ to be smooth and concave on a certain interval of $\alpha$'s. We are able to establish this in the case of equilibrium states for expansive homeomorphisms with specification. The crucial observation, which we exploit in the proof, is the following. Let $\mu = \mu_\varphi$ be an equilibrium state for a potential $\varphi$. Then, applying the result of the previous section, one gets that

$$x \in K_\alpha \iff h_\mu(f,x) = \alpha \iff \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^i(x)) = P(\varphi) - \alpha. \tag{6.16}$$

Therefore the level sets of local entropies are exactly the level sets of limits of ergodic averages of $\varphi$. From the Ergodic Theorem one concludes that only one of these level sets has full measure, while others are of measure 0. We adopt a technique of estimation of the topological entropy of these level sets from [1]. The main idea is the following: we introduce a 1-parameter family of measures such that for each $\alpha$ with $K_\alpha \neq \varnothing$ there is exactly one measure in the family for which $K_\alpha$ has full measure. These measures $\mu_q$ are the equilibrium states for potentials $\varphi_q = q\varphi - P(q\varphi)$. However for the correspondence between levels $\{K_\alpha\}$ and measures $\{\mu_q\}$ we need a parameterization $\alpha(q)$ such that

$$\mu_q(K_{\tilde\alpha}) = \begin{cases} 1, & \text{if } \tilde\alpha = \alpha(q), \\ 0, & \text{if } \tilde\alpha \neq \alpha(q). \end{cases}$$

This parameterization can be given as follows: define $T(q) = P(q\varphi) - qP(\varphi)$, and $\alpha(q) = -T'(q)$ (note that $T$ is $C^1$ by Lemma 6.12). We will establish that

$$h_{top}(f|_{K_\alpha}) = h_{\mu_q}(f),$$

i.e., $\mu_q$ is the measure with maximal metric entropy among all invariant measures $\{\nu\}$ such that $\nu(K_\alpha) = 1$. In order to complete the analysis we have to show that $K_\alpha = \varnothing$ for every $\alpha \notin [\inf_q \alpha(q), \sup_q \alpha(q)]$.

## 6.8 Main Theorem

In this section we state our main result. It is exactly in the form of the corresponding results from [1, 12] on multifractal analysis of local (pointwise) dimensions. We are following the same notation and order of statements. The last statement of our Theorem is analogous to the Remark 5 in [12]. It relates the multifractal spectrum of the local entropies to the spectrum $h_\mu(f,q)$ of the correlation entropies (analogue of the Hentschel–Procaccia spectrum for dimensions $HP(q)$).

**Theorem 6.18.** *Let $f$ be an expansive homeomorphism with specification property of a compact metric space $(X,d)$. Let $\varphi \in \mathcal{V}_f(X)$ and $\mu = \mu_\varphi$ be the corresponding equilibrium state. Then*

*1) For $\mu$-a.e. $x \in X$ the local entropy at $x$ exists and*

$$h_\mu(f,x) = h_\mu(f) = P(\varphi) - \int \varphi \, d\mu.$$

*2) For any $q \in \mathbb{R}$ define*

$$T(q) = P(q\varphi) - qP(\varphi).$$

*Then $T(q)$ is a convex $C^1$ function of $q$. Moreover, $T(0) = h_{top}(f)$, $T(1) = 0$, for every $q \in \mathbb{R}$ one has $T'(q) = \int \varphi d\mu_q - P(\varphi) \leq 0$, where $\mu_q$ is the equilibrium state for $\varphi_q = q\varphi - P(q\varphi)$.*

*3) Put $\alpha(q) = -T'(q)$. Then*

$$\mathcal{E}_E(\alpha(q)) := h_{top}(f|_{K_{\alpha(q)}}) = T(q) + q\alpha(q).$$

*Define*

$$\underline{\alpha} = \inf_q \alpha(q) = \lim_{q\to+\infty} \alpha(q),$$
$$\overline{\alpha} = \sup_q \alpha(q) = \lim_{q\to-\infty} \alpha(q).$$

*Then $K_\alpha = \varnothing$ if $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$. This means that the domain of the multifractal spectrum for local entropies $\alpha \to \mathcal{E}_E(\alpha)$ is the range of the function $q \to -T'(q)$.*

*4) If the equilibrium state $\mu$ for the potential $\varphi$ is not a measure of maximal entropy then the relation between $\mathcal{E}_E$ and $T(q)$ can be written in the following variational form*

$$\mathcal{E}_E(\alpha) = \inf_{q\in\mathbb{R}} (T(q) + q\alpha) \quad \text{for } \alpha \in (\underline{\alpha}, \overline{\alpha}),$$
$$T(q) = \sup_{\alpha\in(\underline{\alpha},\overline{\alpha})} (\mathcal{E}_E(\alpha) - q\alpha) \quad \text{for } q \in \mathbb{R}.$$

*This implies that $\mathcal{E}_E$ is strictly concave and continuously differentiable on $(\underline{\alpha}, \overline{\alpha})$ with the derivative given by $\mathcal{E}_E'(\alpha) = q$, where $q \in \mathbb{R}$ is such that $\alpha = -T'(q)$.*

*5) For every $q \in \mathbb{R}$ the following limits exist*

$$h_\mu(f,q) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} -\frac{1}{n(q-1)} \log \int \mu(\mathcal{B}_n(x,\varepsilon))^{q-1} d\mu,$$
$$R_\mu(f,q) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} -\frac{1}{n(q-1)} \log\Big(\sup_E \sum_{x\in E} \mu(\mathcal{B}_n(x,\varepsilon))^q\Big),$$

*where the supremum is taken over all $(n,\varepsilon)$-separated sets $E$.*

*Moreover,*

$$h_\mu(f,q) = R_\mu(f,q) = -\frac{T(q)}{q-1}.$$

**Proof.** (1) The first statement is a consequence of the Brin-Katok formula for ergodic dynamical systems (Theorem 6.16).

(2) The smoothness and convexity properties of $T$ follow directly from Lemma 6.12. We calculate the derivative of $T$ with respect to $q$. Using the formula from Lemma 6.12 one gets

$$T'(q) = \int \varphi d\mu_q - P(\varphi), \tag{6.17}$$

where $\mu_q$ is the equilibrium state for the potential $q\varphi$ (as well as for the potential $\varphi_q = q\varphi - P(q\varphi)$). The inequality $T'(q) \le 0$ follows from the Variational Principle for $\varphi$.

(3) This statement is taken from [1] where it has not been proven. For the sake of completeness we give the proof here.

Let us first calculate the measure–theoretic entropy of the equilibrium state $\mu_q$. From the variational principle for $\mu_q$ (theorem 6.6) we have

$$\begin{aligned} h_{\mu_q}(f) &= P(\varphi_q) - \int \varphi_q d\mu_q \\ &= 0 + T(q) + qP(\varphi) - q\int \varphi d\mu_q \\ &= T(q) + q\big(P(\varphi) - \int \varphi d\mu_q\big) \\ &= T(q) + q\alpha(q), \end{aligned} \tag{6.18}$$

where $\alpha(q) = -T'(q)$ and we use formula (6.17) for the derivative of $T(q)$.

As we have seen in Lemma 6.17 for any $\alpha$ one has

$$h_\mu(f, x) = \alpha \quad \text{if and only if } \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) = P(\varphi) - \alpha.$$

Let us apply now Lemma 6.17 to the equilibrium state $\mu_q$ corresponding to the potential $q\varphi$. Similarly one gets that for every $\beta$

$$h_{\mu_q}(f, x) = \beta \quad \text{if and only if} \quad q \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) = P(q\varphi) - \beta.$$

Hence one concludes that

$$h_\mu(f, x) = \alpha \quad \text{if and only if} \quad h_{\mu_q}(f, x) = P(q\varphi) - qP(\varphi) + q\alpha.$$

Let now $\alpha = \alpha(q)$. This means that

$$x \in K_{\alpha(q)} \quad \text{if and only if} \quad h_{\mu_q}(f, x) = T(q) + q\alpha(q). \tag{6.19}$$

Combining the results of (6.18) and (6.19) one gets

$$\begin{aligned} h_{\mu_q}(f) &= T(q) + q\alpha(q), \\ h_{\mu_q}(f, x) &= T(q) + q\alpha(q) \quad \text{if and only if } x \in K_{\alpha(q)}. \end{aligned}$$

This means that $h_{\mu_q}(f, x) = h_{\mu_q}(f)$ if and only if $x \in K_\alpha$. Since $\mu_q$ is ergodic, we know from the Brin–Katok formula that $h_{\mu_q}(f, x) = h_{\mu_q}(f)$ for $\mu_q$-a.e. $x \in X$. Hence we conclude that

$$\mu_q(\{x : h_{\mu_q}(f, x) = h_{\mu_q}(f)\}) = \mu_q(K_{\alpha_q}) = 1.$$

Therefore we obtained the desired parameterization of the level sets.

We have to compute the topological entropy of $f$ restricted to $K_{\alpha(q)}$

$$\mathcal{E}_E(\alpha(q)) := h_{top}\left(f|_{K_{\alpha(q)}}\right).$$

Using the properties of the topological entropy from Theorem 6.15, we conclude that

$$\mathcal{E}_E(\alpha(q)) = h_{top}\left(f|_{K_{\alpha(q)}}\right) \geq h_{\mu_q}(f) = T(q) + q\alpha(q)$$

since $\mu_q(K_{\alpha(q)}) = 1$.

We have to prove the opposite inequality. Consider an arbitrary $\lambda > T(q) + q\alpha(q)$. Let $\delta = \lambda - T(q) - q\alpha(q) > 0$. Rewriting the definition of $K_{\alpha(q)}$ in terms of $\mu_q$ and $\varphi_q$, one has

$$\begin{aligned} K_{\alpha(q)} &= \{x \in X : \ h_{\mu_q}(f, x) = h_{\mu_q}(f)\} \\ &= \left\{ x \in X : \ \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi_q(f^i x) = -\left(T(q) + q\alpha(q)\right) \right\}. \end{aligned}$$

For every $x \in K_{\alpha(q)}$ there exists an integer $n(x)$ such that

$$\left|\frac{1}{n}\sum_{i=0}^{n-1}\varphi_q(f^i x) + T(q) + q\alpha(q)\right| \leq \frac{\delta}{2}$$

for all $n \geq n(x)$. For every integer $N$ consider set

$$K_{\alpha(q),N} = \{x \in K_{\alpha(q)} : \ n(x) \leq N\}.$$

Obviously, we have

$$K_{\alpha(q)} = \bigcup_{N\geq 1} K_{\alpha(q),N}, \quad K_{\alpha(q),N} \subseteq K_{\alpha(q),N+1}.$$

Again using the properties of the topological entropy from Theorem 6.15, we conclude that

$$h_{top}(f|_{K_{\alpha(q)}}) = \lim_{N\to\infty} h_{top}(f|_{K_{\alpha(q),N}}).$$

We are going to show that for any $N \in \mathbb{N}$ one has $h_{top}(f|_{K_{\alpha(q),N}}) \leq \lambda$; this in turn will imply $h_{top}(f|_{K_{\alpha(q)}}) \leq \lambda$.

Consider an arbitrary finite cover $\mathcal{U} = \{\mathcal{B}(x_i, \varepsilon/2)\}_{i=1}^{M}$ of $X$ by open balls of radius at most $\varepsilon/2$, with $\varepsilon < \gamma/2$, where $\gamma$ is the expansivity constant for $f$. Together with $\mathcal{U}$ we consider $\tilde{\mathcal{U}}$, an open cover by balls with centers at $x_i$ and radii $\varepsilon$. Let $E = \{y_j\}$ be a maximal $(n, \varepsilon/2)$-separated set in $X$. Define a subset $E'$ of $E$ by choosing those $y_j$ which have a point from $K_{\alpha(q),N}$ close to them, namely

$$E' = \{y_j \in E : \ K_{\alpha(q),N} \cap \mathcal{B}_n(y_j, \varepsilon/2) \neq \varnothing\}.$$

This implies that

$$K_{\alpha(q),N} \subseteq \bigcup_{y_j \in E'} \mathcal{B}_n(y_j, \varepsilon/2).$$

For every $y_j \in E'$ there exists at least one string $\mathbf{U} = U_{i_0}\ldots U_{i_{n-1}}$ from $\mathcal{W}_n(\mathcal{U})$ such that $y_j \in X(U_{i_0}\ldots U_{i_{n-1}})$. It is easy to see that if

$$y_j \in X(U_{i_0,\ldots,i_{n-1}}) = U_{i_0} \cap f^{-1}U_{i_1} \cap \ldots f^{-n+1}U_{i_{n-1}}$$

then

$$\mathcal{B}_n(y_j, \varepsilon/2) \subseteq X(\tilde{U}_{i_0,\ldots,i_{n-1}}) = \tilde{U}_{i_0} \cap f^{-1}\tilde{U}_{i_1} \cap \ldots f^{-n+1}\tilde{U}_{i_{n-1}}.$$

In other words the collection of strings $\tilde{\Gamma} = \{\tilde{U}_{i_0}\ldots U_{i_{n-1}}\}$ covers $K_{\alpha(q),N}$. Therefore,

$$
\begin{aligned}
M(K_{\alpha(q),N},\lambda,\tilde{\mathcal{U}},n) := & \inf_{\substack{\Gamma\subseteq\cup_{k\geq n}\mathcal{W}_k(\tilde{\mathcal{U}})\\ \Gamma \text{ covers } K_{\alpha(q),N}}} \sum_{\mathbf{U}\in\Gamma}\exp(-n(\mathbf{U})\lambda)\\
&\leq \sum_{\tilde{\mathbf{U}}\in\tilde{\Gamma}}\exp(-n(\tilde{\mathbf{U}})\lambda)\\
&= e^{-n\delta}\sum_{\tilde{\mathbf{U}}\in\tilde{\Gamma}}\exp\bigl(-n(T(q)+q\alpha(q))\bigr)\\
&= e^{-n\delta}\sum_{y_j\in E'}\exp\Bigl(-n\bigl(T(q)+q\alpha(q)\bigr)\Bigr)
\end{aligned}
\tag{6.20}
$$

Since the potential $\varphi\in\mathcal{V}_f(X)$, so is $\varphi_q$, and

$$
\left|\sum_{k=0}^{n-1}\varphi_q(f^k(x))-\sum_{k=0}^{n-1}\varphi_q(f^k(y))\right|\leq|q|K
$$

for all $x,y\in X$ with $d_n(x,y)<\varepsilon/2$.

For any $y_j\in E'$ let $x_j$ be an arbitrary point from $K_{\alpha(q),N}\cap\mathcal{B}_n(y_j,\varepsilon/2)$. Then for $n\geq N$ we can continue the estimate (6.20) as follows

$$
\begin{aligned}
&M(K_{\alpha(q),N},\lambda,\tilde{\mathcal{U}},n)\\
&\quad\leq e^{-n\delta+|q|K}\sum_{y_j\in E'}\exp\bigl(-n(T(q)+q\alpha(q))-(S_n\varphi_q)(x_j)+(S_n\varphi_q)(y_j)\bigr)\\
&\quad\leq e^{-n\delta+|q|K}\sum_{y_j\in E'}\exp\bigl(n\delta/2+(S_n\varphi_q)(y_j))\bigr)\\
&\quad\leq e^{-n\delta/2+|q|K}\sum_{y_j\in E'}\exp((S_n\varphi_q)(y_j))\\
&\quad\leq C'e^{-n\delta/2}Z_n(\varphi_q,E).
\end{aligned}
$$

Using the estimates from Lemma (6.13) and the fact that $P(\varphi_q)=0$, we conclude that

$$
M(K_{\alpha(q),N},\lambda,\tilde{\mathcal{U}},n)\leq C''e^{-n\delta/2}.
$$

Hence

$$
M(K_{\alpha(q),N},\lambda,\tilde{\mathcal{U}})=\lim_{n\to\infty}M(K_{\alpha(q),N},\lambda,\tilde{\mathcal{U}},n)=0,
$$

and

$$
M(K_{\alpha(q),N},\lambda)=\lim_{\operatorname{diam}(\mathcal{U})\to 0}M(K_{\alpha(q),N},\lambda,\tilde{\mathcal{U}})=0.
$$

Then by definition of the topological entropy we have

$$
h_{top}(f|_{K_{\alpha(q),N}})\leq\lambda
$$

for all $N$. Hence $h_{top}(f|_{K_{\alpha(q),}})\leq\lambda$ for all $\lambda>T(q)+q\alpha(q)$. This completes the proof the inequality $h_{top}(f|_{K_{\alpha(q),}})\leq T(q)+q\alpha(q)$. The rest of the statement is taken from [20]. It states that we have a complete description of the spectrum for local entropies.

(4) If the equilibrium state for the potential $\varphi$ is not a measure of maximal entropy then it was shown in Lemma 6.12 that $T(q)$ is strictly convex, i.e., the following holds for every $q, q_0 \in \mathbb{R}$, $q \neq q_0$,

$$T(q) > T(q_0) + T'(q_0)(q - q_0).$$

Therefore if $\alpha \in (\underline{\alpha}, \overline{\alpha})$ then there exists $q_0 \in \mathbb{R}$ such that $\alpha = -T'(q_0)$. We have seen that in this case $\mathcal{E}_E(\alpha) = T(q_0) + \alpha q_0$. Using the strict convexity of $T(q)$ we obtain that for $q \in \mathbb{R}$, $q \neq q_0$, and $\alpha = -T'(q_0)$, the following holds

$$\mathcal{E}_E(\alpha) = T(q_0) + \alpha q_0 < T(q) + \alpha q.$$

Hence $\mathcal{E}_E(\alpha) = \inf_{q \in \mathbb{R}} (T(q) + \alpha q)$ for $\alpha \in (\underline{\alpha}, \overline{\alpha})$. In a similar manner one obtains the second relation $T(q) = \sup_{\alpha \in (\underline{\alpha}, \overline{\alpha})} (\mathcal{E}_E(\alpha) - q\alpha)$.

Using the notion of the Legendre transform [16], we can say that actually the functions $T(q)$ and $F(\alpha) := -\mathcal{E}_E(-\alpha)$ form a Legendre pair, i.e., one is the Legendre transform of the other. Therefore the convexity and differentiability of $\mathcal{E}_E$ follow from the properties of Legendre transform. In particular, for $\alpha \in (\underline{\alpha}, \overline{\alpha})$ one has $\mathcal{E}'_E(\alpha) = q$, where $q \in \mathbb{R}$ is such that $\alpha = -T'(q)$.

In the case that $\mu$ is the measure of maximal entropy one has

$$h_\mu(f, x) = h_\mu(f) = h_{top}(f)$$

for all $x \in X$. This means that $\mathcal{E}_E$ is "delta-like" function, i.e.,

$$\mathcal{E}_E(\alpha) = \begin{cases} h_{top}(f), & \text{if } \alpha = h_{top}(f), \\ 0, & \text{otherwise.} \end{cases}$$

This "degenerate" behavior of the multifractal spectrum for the measure of maximal entropy can be successfully exploited. For this see [1], where it has been used for calculations of multifractal spectrum for Lyapunov exponents.

(5) This is an essentially new result. We prove it by means of the standard thermodynamical technique.

Let $q > 1$ and $E$ be an arbitrary $(n, \varepsilon)$-separated set. One has

$$\begin{aligned} \int \mu(\mathcal{B}_n(x, \varepsilon)^{q-1}\, d\mu &\geq \sum_{x_i \in E} \int_{\mathcal{B}_n(x_i, \varepsilon/2)} \mu(\mathcal{B}_n(x, \varepsilon)^{q-1}\, d\mu \\ &\geq \sum_{x_i \in E} \mu(\mathcal{B}_n(x_i, \varepsilon/2))^q, \end{aligned}$$

since $\mathcal{B}_n(x_i, \varepsilon/2) \subseteq \mathcal{B}_n(x, \varepsilon)$ for $x \in \mathcal{B}_n(x_i, \varepsilon/2)$.

Applying the inequality (6.5) and using the fact that $E$ is an $(n, \varepsilon)$-separated set, we get

$$\int \mu(\mathcal{B}_n(x, \varepsilon))^{q-1}\, d\mu \geq \sup_E \left( \sum_{x_i \in E} A^q_{\varepsilon/2} \exp\Big(-qPn + \sum_{j=0}^{n-1} q\varphi(f^j x_i)\Big) \right),$$

where the supremum is taken over all $(n,\varepsilon)$-separated sets. Taking logarithms and applying estimates from Lemma 6.13, we conclude that in the limit

$$h_\mu(f,q) \le R_\mu(f,q) \le \frac{P(q\varphi) - qP(\varphi)}{1-q}.$$

To finish the proof of the statement we have to show the opposite inequalities. We do it in a similar manner. Let now $E$ be a maximal $(n,\varepsilon/2)$-separated set. Then

$$\begin{aligned}\int \mu(\mathcal{B}_n(x,\varepsilon/2))^{q-1} d\mu &\le \sum_{x_i\in E} \int_{\mathcal{B}_n(x_i,\varepsilon/2)} \mu(\mathcal{B}_n(x,\varepsilon/2))^{q-1} d\mu \\ &\le \sum_{x_i\in E} \mu(\mathcal{B}_n(x_i,\varepsilon))^q,\end{aligned}$$

since $\mathcal{B}_n(x,\varepsilon/2) \subseteq \mathcal{B}_n(x_i,\varepsilon)$ holds for every $x \in \mathcal{B}_n(x_i,\varepsilon/2)$.

Again since $E$ is arbitrary $(n,\varepsilon/2)$-separated set and applying the inequality (6.5), we obtain

$$\int \mu(\mathcal{B}_n(x,\varepsilon/2))^{q-1} d\mu \le \sup_E \left( \sum_{x_i\in F} \exp\Big(-qPn + \sum_{j=0}^{n-1} q\varphi(f^j x_i)\Big)\right).$$

Taking logarithms, using estimates from Lemma 6.13 and letting $n \to \infty$, we finally obtain

$$h_\mu(f,q) \ge R_\mu(f,q) \ge \frac{P(q\varphi) - qP(\varphi)}{1-q}.$$

Combining all together we get the final form of the statement in case of $q > 1$. The remaining case $q < 1$ is proved in a similar way. □

## 6.9 Final remarks

A. Consider the "irregular" set

$$\begin{aligned}B &= \{x \in X : \ h_\mu(f,x) \text{ does not exist } \} \\ &= \{x \in X : \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) \text{ does not exist}\}.\end{aligned}$$

We have seen that for the measure of maximal entropy $m_E$ this is an empty set. It was shown in [2] that in a number of cases, the set $B$ is either empty or has full topological entropy and Hausdorff dimension.

B. There exists another way of defining local (pointwise) entropies. Namely, consider an arbitrary finite measurable partition $\xi$ of $X$. We define a local entropy at $x$ with respect to $\xi$ as the following limit (if the limit exists)

$$h_\mu(f,x,\xi) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(\xi^{(n)}(x)),$$

where $\xi^{(n)} = \xi \vee f^{-1}\xi \vee \ldots \vee f^{-n+1}\xi$ and $\xi^{(n)}(x)$ is the element of $\xi^{(n)}$ containing $x$. If $f$ preserves an ergodic measure $\mu$, then $h_\mu(f,x) = h_\mu(f)$ for $\mu$-a.e. $x$ (Shannon-McMillan-Breiman Theorem). We can define a spectrum of local entropies with respect to $\xi$ as follows

$$\mathcal{E}_E(\alpha) = h_{top}(f|_{\{x: h_\mu(f,x,\xi)=\alpha\}}).$$

In case the dynamical system admits some specific partitions (see below), we can show that the multifractal spectrum, defined with respect to these partitions, and the spectrum, defined above, are equal.

Suppose an expanding map $f : X \to X$ admits a finite Markov partition $\mathcal{R} = \{R_1, \ldots, R_M\}$, i.e., a partition with the following properties

1. the closure of every element $R_i$ is a closure of its interior $\mathrm{int}(R_i)$;
2. $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \varnothing$ unless $i = j$;
3. for every $i$, the image $f(R_i)$ is a union of $R_j$'s.

Then the symbolic dynamics, generated by $\mathcal{R}$, is a one-sided subshift of finite type (or a topological Markov chain) with a transfer matrix $A$ such that

$$a_{ij} = 1 \iff \mathrm{int} R_i \cap f^{-1}(\mathrm{int} R_j) \neq \varnothing.$$

Let

$$\Sigma_A^+ = \Big\{\omega = (\omega_0, \omega_1, \ldots) : \ \omega_k \in \{1, \ldots, M\} \text{ and } a_{\omega_k, \omega_{k+1}} = 1 \text{ for all } k \geq 0\Big\},$$

and consider a coding map $\chi : \Sigma_A^+ \to X$ given by

$$\chi(\omega) = \{x \in X : \ f^k x \in R_{\omega_k} \text{ for all } k \geq 0\},$$

where $\omega = (\omega_0, \omega_1, \ldots)$. Then the coding map $\chi$ has the following properties:

1. $\chi$ is a Hölder continuous function;
2. $\chi$ is surjective;
3. $\chi$ is injective on the set of points whose trajectories never hit the boundary;
4. the following diagram, where $\sigma$ is the left shift, commutes

$$\begin{array}{ccc} \Sigma_A^+ & \xrightarrow{\ \sigma\ } & \Sigma_A^+ \\ \chi \big\downarrow & & \big\downarrow \chi \\ J & \xrightarrow{\ f\ } & J \end{array}$$

Let $\varphi$ be a Hölder continuous function on $(X,d)$. The pull back $\psi=\chi_*^{-1}\varphi$, i.e., $\psi(\omega)=\varphi(\xi(\omega))$ is again a Hölder continuous on $(\Sigma_A^+,\rho)$, where $\rho$ is a standard metric on the sequence space. There is a Gibbs measure for $\psi$ on $\Sigma_A^+$, i.e., there are constants $C_1,C_2>0$ such that

$$C_1 \le \frac{\nu_\psi(\omega'\in\Sigma_A^+ : \omega_1=\omega_1',\dots,\omega_n=\omega_n')}{\exp\Big(-nP+\sum_{i=0}^{n-1}\psi(\sigma^i\omega)\Big)} \le C_2$$

for all $\omega$ and $n\ge 1$. Pushing forward the measure $\nu_\psi$ we get a new measure $\mu$ given by $\mu(E)=\nu_\psi(\chi^{-1}E)$ for every measurable set $E\subseteq X$. For the element of the $n$-th iteration of Markov partition $\mathcal{R}$ containing $x$ we will have

$$C \le \frac{\mu(R_{i_1,\dots,i_n}(x))}{\exp\Big(-nP+\sum_{i=0}^{n-1}\varphi(f^i(x))\Big)} \le D.$$

From this it is easy to see that the local entropy with respect to $\mathcal{R}$ at point $x$ exists if and only if there exists limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)).$$

And we obtain exactly the same spectrum for the local entropies with respect to the Markov partition as we had above. Therefore our definition is more general, and coincides with the definition from [1] in the case when the system admits a finite Markov partition.

C. The results of this chapter can be extended to the case of expansive endomorphisms (i.e., non-invertible maps) with the specification property. They are defined exactly in the same way as the expansive homeomorphisms with specification except that the set $\mathbb{Z}$ in (6.1) is substituted by $\mathbb{N}$ (positive expansiveness). The characteristic property of the equilibrium states (Theorem 6.10) remains valid [18]. Therefore our analysis works without any modifications.

In the case of expansive homeomorphisms we can give another definition of local entropies. Namely, for any $n\ge 1$ define

$$\mathcal{B}_n^{\pm}(x,\varepsilon)=\{y\in X:\ d(f^i(x),f^i(y))<\varepsilon \quad \text{for all} \quad i=-n+1,\dots,n-1\},$$

and

$$\underline{h}_\mu^{\pm}(f,x)=\lim_{\varepsilon\to 0}\liminf_{n\to\infty}-\frac{1}{2n-1}\log\mu(\mathcal{B}_n^{\pm}(x,\varepsilon)),$$
$$\overline{h}_\mu^{\pm}(f,x)=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}-\frac{1}{2n-1}\log\mu(\mathcal{B}_n^{\pm}(x,\varepsilon)).$$

Then the level sets of these local entropies will be in one-to-one correspondence with the level sets of two-sided ergodic averages of $\varphi$

$$\lim_{n\to\infty}\frac{1}{2n-1}\sum_{k=-n+1}^{n-1}\varphi(f^k(x)).$$

The level sets of two-sided and one-sided ergodic averages of $\varphi$ can be different. However they have the same topological entropy with respect to $f$. Therefore the multifractal spectrum based on $h^{\pm}_{\mu}(f,x)$ will be the same.

D. It is important to stress that smooth transformations with specification property do not necessarily admit finite Markov partitions. Existence of Markov partitions was a crucial in the previous works on multifractal analysis of dynamical systems. Let us illustrate our claim with some examples.

In [3], A. Blokh showed that topologically mixing continuous transformations of the unit interval have the specification property. Topological mixing is not exceptional even for non-hyperbolic interval maps. For example, M. Jakobson [8] proved that in the logistic family $f_r(x) = rx(1-x)$ topological mixing is not untypical. In particular, it holds for a set of parameters of positive Lebesgue measure. Of course, logistic maps are not necessarily expansive, and not much is known about the thermodynamic properties of the logistic family. Hence, the results of this chapter are not applicable. However, using a different approach, some results of multifractal analysis for systems with specification property have been obtained in [21].

It is also possible to construct examples of smooth dynamical systems with specification. Let us start with an example of a piecewise expanding interval map with specification property, which does not admit a finite Markov partition. Consider the so-called $\beta$-shifts, which for $\beta > 1$ are given by

$$f_\beta(x) = \beta x \mod 1.$$

One can show [19] that there are exist values of the parameter $\beta$ such that $f_\beta$ has the specification property, but does not admit a finite Markov partition.

Consider symbolic dynamical systems. Let $\Sigma = \Omega^{\mathbb{Z}}$, where $\Omega$ is a finite alphabet, $\sigma : \Sigma \to \Sigma$ be the left shift. By definition, a *subshift* $S$ is a closed shift-invariant subset of $\Sigma$. Analogs of systems with Markov partitions are the so-called *subshifts of finite type* considered above. Under mild mixing assumptions, subshifts of finite type have the specification property. One can easily construct examples of subshifts with the specification property, which are not subshifts of finite type. For example, one can use $\beta$-shifts, discussed above, to generate such subshifts. Using an arbitrary subshift, one can construct a closed invariant subset $K$ of an appropriate hyperbolic set $\Lambda$ of a diffeomorphism $f$. In this way, by choosing a subshift with specification which is not a subshift of finite type, we obtain a smooth dynamical system $f : K \to K$ with the specification property, but without a finite Markov partition.

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# Chapter 7

# Intermittency and weak Gibbs states

We show that the natural invariant state for Manneville-Pomeau maps can be characterized as a weakly Gibbsian state. In this way we make a connection between the study of intermittency via non-uniformly expanding maps and the thermodynamic formalism for non-uniformly convergent interactions, which is motivated by recent developments in statistical mechanics.

In this chapter we assume the reader to be familiar with the standard notions of statistical mechanics.

## 7.1 Introduction

In this chapter we connect the notion of weakly Gibbsian state, as has recently emerged from the statistical mechanical study of certain lattice spin systems, with the concept of intermittency, as modeled by Manneville-Pomeau maps.

Weakly Gibbsian states were introduced by R.L. Dobrushin in his last conference talk in Renkum, The Netherlands in 1995. What was sought for was a Gibbsian restoration of certain physically relevant examples of non-Gibbsian states. A first part of the Dobrushin program has been recently completed in [21] where it is shown that essentially all restrictions to a sublattice of the low temperature phases in the realm of the Pirogov-Sinai theory for lattice spin systems, are weakly Gibbsian. The typical scenario is the occurrence of a 'configuration dependent range of the interaction', which implies that the relative energies are no longer uniformly bounded (as is the case for the usual Gibbsian set-up) but can be unbounded as dictated by configuration dependent length scales. This divides the set of lattice spin configurations in two disjoint sets: the 'good' ones for which the effective interaction is short range, and the 'bad' ones, for which the total interaction is di-

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verging. Instead of introducing the somewhat abstract formalism defining weakly Gibbsian states, we refer to [22] for general definitions and properties and we only underline the above via a concrete and, for our purposes, illustrative example.

### 7.1.1 Example of weakly Gibbsian state

Consider the standard ferromagnetic Ising model on the square lattice $\mathbb{Z}^2$ with the usual nearest neighbor interactions. The finite volume Gibbs measure $\mu_{\beta,n}$ on a box $\Lambda_n$ with plus boundary conditions is defined on the infinite volume Ising configuration space $\Omega = \{+1,-1\}^{\mathbb{Z}^2}$ via

$$\mu_{\beta,n}(\sigma) = \frac{I[\sigma = \bar{1} \text{ on } \Lambda_n^c]}{Z_n(\beta)} \exp[\beta \sum_{\langle xy\rangle \cap \Lambda_n \neq \varnothing} \sigma_x \sigma_y]$$

where $\Lambda_n = [-n,n]^2 \cap \mathbb{Z}^2$ is a finite box, $I[\sigma = \bar{1} \text{ on } \Lambda_n^c]$ is the indicator of the event that $\sigma_z = +1$ for all $z \in \Lambda_n^c = \mathbb{Z}^2 \setminus \Lambda_n$, $\beta > 0$ stands for the inverse temperature, $Z_n(\beta)$ is the normalizing partition function and the sum in the exponent is over all nearest neighbor pairs $\langle xy\rangle$ at least one of which is in the box $\Lambda_n$. It is well known that the weak limit $\lim_n \mu_{\beta,n} = \mu_\beta$ exists for all $\beta$. This limit is a translation invariant Gibbs measure at inverse temperature $\beta$ for the formal Hamiltonian

$$H(\sigma) = -\sum_{\langle xy\rangle} \sigma_x \sigma_y.$$

Of course, this sum is only well defined in terms of the corresponding relative energies

$$H(\sigma) - H(\eta) = H_\Lambda(\sigma) = -\sum_{\langle xy\rangle \cap \Lambda \neq \varnothing} (\sigma_x\sigma_y - \eta_x\eta_y)$$

defined for a finite region $\Lambda \subseteq \mathbb{Z}^2$ and $\eta = \sigma$ on $\Lambda^c$ ($\eta$ coincides with $\sigma$ outside a finite volume). That $\mu_\beta$ is a Gibbs measure for $H$ means that its conditional probabilities are described via these relative energies as for example in

$$\mu_\beta(\sigma_0 | \sigma_x, x \neq 0) = \frac{1}{1 + \exp[-2\beta \sum_{\langle x0\rangle} \sigma_0 \sigma_x]}.$$

In particular, the measure $\mu_\beta$ admits a continuous version of its conditional probabilities.

We are interested in the restriction $\nu_\beta$ of this infinite volume probability measure $\mu_\beta$ to a lattice line (to be identified with $\mathbb{Z}$), say one which contains the origin. It was proven in [36, 31] that at low temperatures ($\beta$ sufficiently large) $\nu_\beta$ is not Gibbsian, i.e., does not admit a continuous version of its conditional probabilities. It was however realized by Dobrushin that $\nu_\beta$ remains weakly Gibbsian. This means the following. There exists a translation invariant tail-set $K \subseteq \{+1,-1\}^{\mathbb{Z}}$ of 'good' one-dimensional lattice spin configurations which has full measure ($\nu_\beta(K) = 1$) and for which one can find a translation invariant interaction potential $(U_A)$, which is absolutely summable on $K$ and is compatible with $\nu_\beta$, i.e., the interaction potential is a collection of functions $U_A : \{+1, -1\}^A \to \mathbb{R}$ parameterized by the finite subsets $A$ of $\mathbb{Z}$, for which

$$\sum_{A \ni 0} |U_A(\xi)| < \infty, \quad \xi \in K$$

(absolute convergence) and for which the Dobrushin-Lanford-Ruelle (DLR) equations with respect to $\nu_\beta$ are satisfied:

$$\int f(\xi) d\nu_\beta(\xi) = \int \sum_{\omega \in \{+1,-1\}^V} \frac{1}{Z_V^U(\xi_{V^c})} f(\omega_V \xi_{V^c}) \exp\Big(-\sum_{A \cap V \neq \varnothing} U_A(\omega_V \xi_{V^c})\Big) d\nu_\beta(\xi) \tag{7.1}$$

for all continuous functions $f$ on $\{+1, -1\}^{\mathbb{Z}}$ for all finite $V \subseteq \mathbb{Z}$, and where $\omega_V \xi_{V^c}$ is a configuration, which coincides with $\omega$ on $V$, and with $\xi$ on $V^c$.

In this sense, $\nu_\beta$ is weakly Gibbsian for the formal Hamiltonian

$$H(\xi) = \sum_A U_A(\xi),$$

but now, even the local Hamiltonians

$$H_V^U(\xi) = \sum_{A \cap V \neq \varnothing} U_A(\xi)$$

are only well defined for $\xi \in K$.

The proof of this result (i.e., an existence of a tail-set $K$) was given in [24, 23, 6, 7] with a more general version in [21]. It turns out that one can choose the potential $(U_A)$ so that it is non-vanishing only for $A$ a lattice interval. In particular, one shows that for every $\xi \in K$ there is a (configuration dependent length) $\ell(\xi) < +\infty$ for which

$$|U_{[0,k]}(\xi)| \leq c_1 I[k \leq \ell(\xi)] + c_2 \exp[-c_3 k]$$

for all $k > 0$ and, where the finite constants $c_1, c_2, c_3$ depend on $\beta$. In other words, the potential starts decaying only after a 'random' distance which is itself function of the configuration. That is the meaning of saying that the interaction is effectively short ranged with a 'configuration dependent interaction range.' In the model, this range $\ell(\xi)$ measures the distance to the right of the origin after which the proportion of $+1$-spins to the right of the origin becomes for ever larger than a given (large) amount. It is this structure of the interaction that reminds us of the phenomenon of intermittency in the theory of dynamical systems.

### 7.1.2 Intermittency

Since the beginning of the 80's intermittency has been widely studied as a common phenomenon in the transition to turbulence, [1]. While it is difficult to give a good definition, its simplest manifestation is probably the occurrence of randomly spread bursts or fluctuations happening between periods where the system undergoes a limit cycle or periodic motion. While varying some control parameter, the average frequency of these fluctuations becomes larger and larger. Here we will not discuss the nature of this intermittent regime except for investigating some Gibbsian aspects of the steady state for some model systems.

To see what we have in mind, it is best to start from so called (uniformly) expanding interval maps. Under some additional smoothness conditions, there is a unique ergodic absolutely continuous time-invariant measure. Its density is a continuous function bounded away from zero. The standard Gibbs formalism can be applied and an exponentially decaying interaction can be identified with which this invariant measure is compatible. Imagine now what happens if an indifferent fixed point appears. In the neighborhood of this point the expansion of the map shrinks to zero, because the derivative in the indifferent fixed point is equal to one. This non-uniformity in the expansion has as a consequence that the system can stay for longer times in the neighborhood of this fixed point before it is expelled to a region where the map is again truly expanding. These fluctuations, i.e., long periods near the fixed point, are rare but are nevertheless responsible for breaking the uniform convergence of an associated interaction potential. It is this feature that we study here.

We start in the next section with the introduction of the simplest models. Section 3 is devoted to the presentation of our main result: the weakly Gibbsian character of the absolutely continuous invariant measure.

## 7.2 Model: Interval maps with indifferent fixed points

### 7.2.1 Model

We study the following class of non-uniformly expanding interval maps.

**Definition 7.1.** *We say that $T : [0,1] \to [0,1]$ is a Manneville-Pomeau type map (or, shortly, a MP map) if*

*(i) there exists a $p > 0$ such that $T|_{(0,p)}$ and $T|_{(p,1)}$ are strictly monotone, continuous and $T(0,p) = T(p,1) = (0,1)$;*

*(ii) the branches $T|_{(0,p)}$ and $T|_{(p,1]}$ are $C^2$;*

*(iii) $T'(x) > 1$ for all $x > 0$, $x \neq p$, and $T'(x) \geq \lambda > 1$ for $x \in (p,1)$.*

*(iv) $T$ has the following asymptotic behavior when $x \to 0_+$:*

$$T(x) = x + Cx^{1+\alpha}(1 + u(x))$$

*for some constants $C > 0$, $\alpha \in (0,1)$, and $u$ is a $C^2$ function such that*

$$\lim_{x \to 0_+} u'(x) = \lim_{x \to 0_+} u''(x) = 0.$$

As an example we can consider the original Manneville-Pomeau map itself:

$$T(x) = x + x^{1+\alpha} \mod 1, \quad \alpha \in (0,1).$$

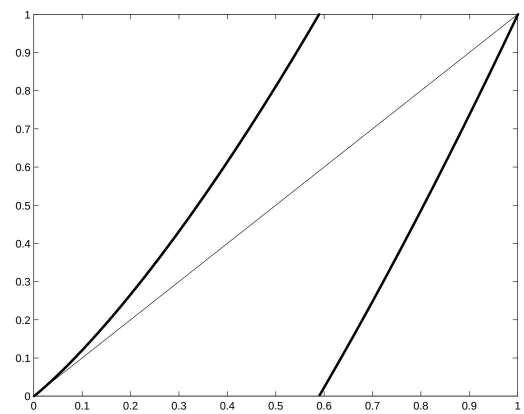


Figure 7.1: A Manneville-Pomeau type map.

*Remark.* It is easy to see that $(iii)$ and $(iv)$ imply that 0 is a unique indifferent fixed point.

### 7.2.2 Absolutely continuous invariant measure, ergodic properties

Pianigiani [26] established the existence of an absolutely continuous invariant measures for MP maps. The constructed absolutely continuous invariant measure $\mu$ for the MP map $T$ is a Sinai-Ruelle-Bowen measure: for almost every $x \in [0,1]$ with respect to the Lebesgue measure, one has the weak convergence

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i(x)} \to \mu,$$

where $\delta_y$ is the Dirac measure at $y$. Thaler [33, 34] has proven the following estimates on the density $h(x) = \dfrac{d\mu}{dx}$ for MP maps: there exist constants $C_*, C^* \in$

$(0,\infty)$ such that

$$\frac{C_*}{x^\alpha} < h(x) < \frac{C^*}{x^\alpha} \quad \text{for all } x > 0. \tag{7.2}$$

It is also not very difficult to see that the dynamical system $([0,1],\mu,T)$ is exact:

$$\lim_{n\to\infty} \mu(T^n(A)) = 1$$

for all measurable sets $A$ with $\mu(A) > 0$, implying ergodicity and mixing.

Determining a rate of mixing (or, decay of correlations) for the MP-type maps attracted a lot of attention. This problem has been studied in [14, 20, 13, 39]. It turns out that for the Manneville-Pomeau type maps one has a polynomial decay of correlations: for sufficiently smooth $f,g$ (say, Hölder continuous)

$$|\rho(n)| = \left| \int f(x)g(T^n(x))d\mu - \int f(x)d\mu \int g(x)d\mu \right| = O(n^{-\frac{1}{\alpha}+1}).$$

It should be mentioned that there are several other possibilities for piecewise monotone interval maps with indifferent fixed points to have a finite absolutely continuous invariant measure even with bounded density, e.g. [16, 41]. What is important is that the presence of an indifferent periodic point (i.e., $T^p(x) = x$ and $|(T^p)'(x)| = 1$ for some $x \in [0,1]$ and $p \geq 1$) should be compensated by some singularities of the first or second derivative of $T$, because if the map is $C^2$, only infinite absolutely continuous $T-$invariant measures exist, see [4].

### 7.2.3 Thermodynamic formalism

Formally, an MP map $T$ is not a continuous transformation of a compact metric space – the interval $[0,1]$. However, we can make one from $T$. This is done by doubling the point of discontinuity $p$, i.e., substituting it by two points $p_-$ and $p_+$, such that $p_- < p_+$, and putting $T(p_-) = \lim_{x\uparrow p} T(x)$ and $T(p_+) = \lim_{x\downarrow p} T(x)$. We repeat the procedure with all the preimages of $p_-$ and $p_+$. In this way we obtain an 'enlarged' space $X$, which is totally ordered and order complete. Moreover, $X$ is a compact space. In this new space $X$, the intervals $\bar{I}_i$ form a partition. $X$ has points which are isolated from one side, but there are no completely isolated points. Since at most a countable number of points are affected by this operation, and since we are studying measures that are absolutely continuous with respect to the Lebesgue measure, the described modifications take place on a set of measure 0, and are therefore irrelevant from a measure-theoretic point of view. Note also, that this operation makes the coding map $\pi : \{0,1\}^{\mathbb{Z}_+} \to X$, given by

$$\pi(\omega_0\omega_1\omega_2\ldots) = \bar{I}_{\omega_0} \cap T^{-1}\bar{I}_{\omega_1} \cap T^{-2}\bar{I}_{\omega_2} \cap \ldots,$$

a homeomorphism. The coding $\pi$ conjugates $T$ with the left shift $\sigma$ on $\Sigma = \{0,1\}^{\mathbb{Z}_+}$, i.e., $T\circ\pi = \pi\circ\sigma$. For the details see [12] and [16, Appendix A.5].

Consider the function $\varphi = -\log|T'|$, where $T'(x)$ is the left or right derivative of $T$ at $x$ if $x$ is isolated from the right or left respectively. The topological pressure of $\varphi$ is

$$P(\varphi) = \sup_{\nu}\Big(h_\nu(T) + \int \varphi \, d\nu\Big),$$

where the supremum is taken over all $T$-invariant measures and $h_\nu(T)$ is the measure-theoretic entropy (Kolmogorov-Sinai entropy) (see e.g. the variational principle in [37]). The absolutely continuous invariant measure $\mu$ is an equilibrium state for $\varphi$, i.e.,

$$P(\varphi) = h_\mu(T) + \int \varphi \, d\mu. \tag{7.3}$$

Since $\mu$ is an absolutely continuous invariant measure, the measure-theoretic entropy (Kolmogorov-Sinai entropy) is given by Rokhlin's formula [19]:

$$h_\mu(T) = \int \log|T'| \, d\mu = -\int \varphi \, d\mu.$$

and hence $P(\varphi) = 0$. However $\mu$ is not the only equilibrium state. The Dirac measure at 0, which we denote by $\delta_0$, satisfies (7.3) as well. Hence every measure from the convex hull of $\mu$ and $\delta_0$:

$$\mathcal{A} = \big\{ t\mu + (1-t)\delta_0 | \; t \in [0,1] \big\},$$

is an equilibrium state. There are no other equilibrium states for $\varphi$.

Non-uniqueness of the equilibrium states for $\varphi$ results in a singular behavior of the pressure function $P(q\varphi)$, $q \in \mathbb{R}$. Combining the results from [28, p. 511] and [35, Theorem 3.6] we obtain the following statement on the type of phase transition.

**Theorem 7.2.** *Let $T$ be an MP map. The pressure function $P(q\varphi)$ is continuous, convex and non-increasing. Moreover, $P(q\varphi) = 0$ for $q \geq 1$, $P(q\varphi) > 0$ for $q < 1$, and $P(q\varphi)$ is a real-analytic function of $q$ for $q < 1$. At the critical point one has the following asymptotics*

$$\frac{P(q\varphi)}{1-q} \to h_\mu(T) \;\; \textit{as } q \nearrow 1.$$

It is known that for the expanding interval maps the so-called *multifractal formalism* is valid, see e.g. [25]. As the next step in understanding the thermodynamic properties of the interval maps with indifferent fixed points, it would be important to understand if the multifractal formalism is still valid for such systems. A partial multifractal analysis of local dimensions and Lyapunov exponents has been performed for MP maps in [27]. Using a different approach complete description of the multifractal spectra for Lyapunov exponents and local entropies has been obtained [32]. It was shown in [32], that due to the phase transition, these multifractal spectra have finite smoothness $C^1$, while in the case of expanding interval maps these spectra are real-analytic.

## 7.3 Main results: Gibbs properties of MP maps

### 7.3.1 Unbounded distortion

Consider a piecewise monotonic map $T$ of the unit interval $I$. Denote by $\{I_k\}$ the intervals of monotonicity of $T$. Assume that $T$ can be continued upto $C^2$ diffeomorphism $T_k$ on the closure of $I_k$, and $T_k(\bar{I}_k) = [0,1]$. Assume also that $T$ is expanding, i.e., there exists $\lambda > 1$ such that $|T'(x)| \geq \lambda$ for all $x \in I_k$.

Such map $T$ admits an absolutely continuous invariant measure $\mu$, whose density $h$ is a continuous function bounded away from 0, see [3]. This measure $\mu$ has the following useful property: there exists a constant $C > 1$ such that for all $x \in [0,1]$ and every $n \geq 1$ one has

$$\frac{1}{C} \leq \frac{\mu\Big(I_{i_1,\dots,i_n}(x)\Big)}{\exp\Big(\sum_{k=0}^{n-1} \varphi\big(T^k(x)\big)\Big)} \leq C \tag{7.4}$$

where $\varphi = -\log|T'|$ and $I_{i_1,\dots,i_n}(x) = I_{i_1} \cap T^{-1} I_2 \cap \ldots \cap T^{-n+1} I_{i_n}$ is that interval of monotonicity for $T^n$, which contains $x$.

This property (7.4), which we call Bowen's boundness property, is often taken as a definition of a Gibbs state in dynamical systems. Indeed, the inequalities in (7.4) can be derived from standard definitions of Gibbs state (see [16] for details). We can also obtain these inequalities from the properties of expanding maps and absolutely continuous measures directly.

First of all, expanding interval maps have the so-called bounded distortion property: there exists some constant $C > 0$ such that

$$\frac{1}{C} \leq \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq C \tag{7.5}$$

for all $x, y \in I_{i_1,\dots,i_n}$. Secondly, $T^n\big(I_{i_1,\dots,i_n}\big) = (0,1)$. Thus, from the Mean Value theorem we conclude that there exists a point $x^* \in I_{i_1,\dots,i_n}$ such that

$$1 = m\big(T^n I_{i_1,\dots,i_n}\big) = |(T^n)'(x^*)| m(I_{i_1,\dots,i_n}).$$

Taking into account (7.5) and the fact that the density $h$ is a continuous function bounded away from 0, we obtain (7.4).

Manneville-Pomeau maps do not have the bounded distortion property. This is most clearly seen on the leftmost interval of monotonicity of $T^n$. This interval contains zero, therefore $\inf_{x \in I_{0,\dots,0}} |(T^n)'(x)| = 1$. On the other hand, $\sup_{x \in I_{0,\dots,0}} |(T^n)'(x)| \geq 1/m(I_{0,\dots,0}) \to \infty$ as $n \to \infty$. As a result the ratio

$$\frac{\mu\Big(I_{i_1,\dots,i_n}(x)\Big)}{\exp\Big(\sum_{i=0}^{n-1} \varphi\big(T^k(x)\big)\Big)}$$

is not uniformly bounded in $x$ and $n$. However one can find bounds from above and below which are polynomial in $n$ and uniform in $x$. This observation (i.e., the violation of (7.4) for the absolutely continuous measure in the case of MP maps) motivated Yuri [40] to call $\mu$ weakly Gibbs. We wish to show that $\mu$ is indeed a weakly Gibbs measure in the sense of [22, 21].

### 7.3.2 Weakly Gibbsian measures for MP maps

Often the natural invariant measures for dynamical systems (such as SRB measures) can be connected with the Gibbs states as they appear in mathematical statistical mechanics. There are two possible ways to do so. The first approach is to follow the prescription of Capocaccia which requires the existence of a well-defined relative energy under 'local' transformations. This approach is quite general; it, for example, allows to establish thermodynamic formalism even for some systems without Markov partitions, e.g., expansive homeomorphisms with the specification property [30, 11]. We will follow this approach in section 7.3.2.

The second and more traditional way is to use an appropriate (Markov) partition and the symbolic dynamics, and discuss the Gibbsian aspects of the image measure as is usual for lattice spin systems. This approach, for example, allows us to study the properties of the conditional probabilities on finite sets (boxes), given the configuration outside. This will be done in section 7.3.2. In particular, in theorem 7.8 we will construct a symbolic interaction potential for MP maps and show that it is not absolutely convergent, but is convergent on a set of measure 1. Moreover similar to the projection of the Ising model discussed in the introduction, we will establish that for almost every configuration $\omega$ one can find a configuration dependent length $l(\omega)$ after which the interaction potential decays exponentially. Finally, in theorem 7.9 we will relate the distribution of this length $l(\omega)$ with the decay of correlations for the MP maps.

#### Gibbs property and multipliers

We will use a definition of a Gibbs state introduced by Capocaccia in [5], see also [30, 10, 9, 16]. According to this definition, a measure $\mu$ is called Gibbs if the result $\tau_*\mu$ of an action by an arbitrary conjugating homeomorphism (defined below) is absolutely continuous with respect to $\mu$ and the corresponding Radon-Nykodim derivative, which depends on $\tau$ and is called a multiplier for $\tau$, has certain properties. The DLR equations (7.1) can then be rewritten in terms of these multipliers. We recall some definitions from [5, 30, 11].

**Definition 7.3.** *A continuous transformation $T$ of a compact metric space $(X, d)$ is called expansive if there exists $\gamma > 0$ such that if $d(T^k(x), T^k(y)) < \gamma$ for all $k \geq 0$ then $x = y$. Two points $x, y \in X$ are called conjugated if $d(T^k(x), T^k(y)) \to 0$ as $k \to \infty$. Two points $x, y \in X$ are called $n$-conjugated if $T^n(x) = T^n(y)$.*

Clearly, MP maps are expansive. Also, it is easy to see, that if $T$ is an expansive endomorphism, and two points $x, y \in X$ are conjugated, then they are $n$-conjugated for some $n \in \mathbb{N}$. Therefore, points $x$ and $y$ are conjugated if and only if their symbolic representations $\omega = (\omega_0, \omega_1, \dots)$, $\omega' = (\omega'_0, \omega'_1, \dots)$ coincide starting from a certain place, i.e., there exists $n \in N$ such that

$$\omega_k = \omega'_k \text{ for } k \geq n.$$

**Definition 7.4.** *A homeomorphism $\tau : U \to X$, defined on a closed set $U$, $U \subseteq X$, is called* **conjugating** *if $x$ and $\tau(x)$ are conjugated for every $x \in U$.*

*Remark.* Generally conjugating homeomorphisms do not form a group, but a **pseudogroup**: composition of two conjugating homeomorphisms $\tau'$, $\tau''$, defined on $U'$ and $U''$ respectively, can be defined provided $U = (\tau')^{-1}(U'' \cap \tau'(U'))$ is not empty. In this case $\tau = \tau'' \circ \tau'$ is a conjugating homeomorphism defined on $U$. We will use this observation latter when we discuss the cocycle property of multipliers.

If two points $x$ and $y$ are conjugated, then there is a unique germ of a conjugating homeomorphism mapping a neighborhood of $x$ into a neighborhood of $y$ [5, 10]. Those germs form a groupoid [38].

We are going to describe a set of conjugating homeomorphisms $\mathcal{E}$, [16], for MP maps $T$, using the fact that $T : X \to X$ is topologically conjugated to a one-sided shift $\sigma : \Sigma \to \Sigma$, $\Sigma = \{0,1\}^{\mathbb{Z}_+}$, by a coding map $\pi : \Sigma \to X$. By definition $\mathcal{E} = \cup_n \mathcal{E}_n$, where $\mathcal{E}_n$ is defined as follows. We say that $\tau \in \mathcal{E}_n$, $n \geq 1$, if and only if

1) there exist $(i_0, \ldots, i_{n-1}), (j_0, \ldots, j_{n-1}) \in \{0,1\}^n$ such that

$$\tau : \bar{I}_{i_0,\ldots,i_{n-1}} \to \bar{I}_{j_0,\ldots,j_{n-1}};$$

2) for every point $x \in \bar{I}_{i_0,\ldots,i_{n-1}}$ with the symbolic representation

$$\omega = \pi^{-1}(x) = (i_0, \ldots, i_{n-1}, \omega_n, \omega_{n+1}, \ldots)$$

the image $\tau(x) = y$ has a symbolic representation

$$\omega' = \pi^{-1}(y) = (j_0, \ldots, j_{n-1}, \omega_n, \omega_{n+1}, \ldots).$$

It means, that $\tau$ alters the first $n$ symbols in the symbolic representation of $x$. In a compact form $\tau$ can be written as follows

$$\tau(x) = \pi\big((j_0, \ldots, j_{n-1}) \vee \sigma^n(\pi^{-1}(x))\big),$$

where $(j_0, \ldots, j_{n-1}) \vee \sigma^n(\pi^{-1}(x))$ is the concatenation of strings $(j_0, \ldots, j_{n-1})$ and $\sigma^n(\pi^{-1}(x))$.

Now, we give the definition of (weakly) Gibbs states.

**Definition 7.5.** *Suppose $(X, d)$ is a compact metric space, $T : X \to X$ is a continuous expansive transformation.*

*1) A family of non-negative functions $\{R_\tau\}$, indexed by all conjugating homeomorphisms $\{\tau | \tau : U \to \tau(U)\}$, is called a* **family of multipliers** *if the following cocycle relation*

$$R_{\tau''}\big(R_{\tau'} \circ \tau''^{-1}\big) = R_{\tau'' \circ \tau'} \tag{7.6}$$

*holds on $U = (\tau')^{-1}(\tau'(U') \cap U'')$, whenever $U$ is not empty.*

*2) A measure $\mu$ is called* **weakly Gibbs** *for the family of multipliers $\{R_\tau\}$ if for every conjugating homeomorphism $\tau : U \to \tau(U)$, the push-forward $\tau_*(\mu|_U)$ is absolutely continuous with respect to $\mu|_{\tau(U)}$ and*

$$\frac{d\tau_*(\mu|_U)}{d\mu|_{\tau(U)}} = R_\tau. \tag{7.7}$$

*3) A measure $\mu$ is called* **Gibbs** *if it is weakly Gibbs for some family of positive and continuous multipliers $\{R_\tau\}$.*

*4) A function $\varphi : X \to \mathbb{R}$ is called a* **dynamical potential** *for the measure $\mu$ if $\mu$ is (weakly) Gibbs and*

$$R_\tau = \exp\Big(\sum_{k=0}^{\infty} \varphi(T^k \circ \tau^{-1}(x)) - \varphi(T^k(x))\Big). \tag{7.8}$$

*Remark.* a) Condition (7.7) is equivalent to the requirement

$$\int f \circ \tau d\mu = \int f R_\tau d\mu$$

for all continuous $f$ supported on $\tau(U)$.

b) If $\tau$ is a conjugating homeomorphism, so is $\tau^{-1}$. It is easy to see that due to expansiveness of $T$ two conjugated points $x$ and $y$ are $n$-conjugated for some $n \geq 0$, i.e., $T^n(x) = T^n(y)$. Therefore the sum in (7.8) is actually finite.

c) It is also easy to check that any family of functions $R_\tau$, obtained from (7.8), is a family of multipliers in the sense of (7.6).

Therefore in order to decide whether a given measure $\nu$ is Gibbs or not, we have to understand what happens to $\nu$ under the action of all possible conjugating homeomorphisms from $\mathcal{E}$. This seems to be an enormous task. Nevertheless the problem becomes much easier if we can relate the measure $\nu$ to some transfer operator.

Suppose we are given some non-negative Borel bounded function $\psi : X \to \mathbb{R}$. Define the corresponding (Ruelle's) transfer operator $\mathcal{L}_\psi$, acting on bounded Borel measurable functions , as follows

$$\mathcal{L}_\psi f(x) = \sum_{y:T(y)=x} \psi(y) f(y).$$

By induction,

$$\mathcal{L}_\psi^n f(x) = \sum_{y:T^n(y)=x} \psi(y)\psi(T(y)) \dots \psi(T^{n-1}(y)) f(y).$$

Define also an adjoint operator $\mathcal{L}_\psi^*$, acting on Borel measures, by requiring that the equality

$$\int \mathcal{L}_\psi f \, d\nu = \int f \, d\mathcal{L}_\psi^* \nu,$$

holds for all $f$. The following is a corollary of a theorem by Ruelle [30].

**Theorem 7.6.** *Suppose $T : X \to X$ is a Manneville-Pomeau map, and $\nu$ is a Borel measure (not necessarily $T$–invariant) such that*

$$\mathcal{L}_{\psi}^{*}\nu = \nu,$$

*and $\psi$ is positive $\nu$-almost everywhere. Then for every $\tau \in \mathcal{E}$ the following holds*

$$\tau_*(\nu|_U) \ll \nu|_{\tau(U)} \quad \text{and} \quad R_\tau := \frac{d\tau_*\nu}{d\nu} = \frac{\prod_{k=0}^{\infty}\psi\big(T^k \circ \tau^{-1}(x)\big)}{\prod_{k=0}^{\infty}\psi\big(T^k(x)\big)}. \tag{7.9}$$

*Proof.* Let $\tau \in \mathcal{E}$. Then there exists $n \geq 1$ and $(i_0, \dots, i_{n-1})$, $(j_0, \dots, j_{n-1})$ such that

$$\tau : \bar{I}_{i_0,\dots,i_{n-1}} \to \bar{I}_{j_0,\dots,j_{n-1}}.$$

Consider an arbitrary bounded Borel function $f$ vanishing outside $\bar{I}_{j_0,\dots,j_{n-1}}$. Then

$$\mathcal{L}_{\psi}^{n}(f \circ \tau)(x) = \prod_{k=0}^{n-1}\psi\big(T^k(y)\big)f(\tau(y)),$$

where $y \in \bar{I}_{i_0,\dots,i_{n-1}}$ is such that $T^n(y) = x$. Or equivalently,

$$\mathcal{L}_{\psi}^{n}(f \circ \tau)(x) = \prod_{k=0}^{n-1}\psi\big(T^k(\tau^{-1}(z))\big)f(z),$$

where $z \in \bar{I}_{j_0,\dots,j_{n-1}}$ is such that $T^n(z) = x$. Therefore,

$$\begin{aligned}
\mathcal{L}_{\psi}^{n}\big(f \circ \tau\big)(x) &= \prod_{k=0}^{n-1}\psi\big(T^k(z)\big)\frac{\prod_{k=0}^{n-1}\psi\big(T^k \circ \tau^{-1}(z)\big)}{\prod_{k=0}^{n-1}\psi\big(T^k(z)\big)}f(z)\\
&= \left(\mathcal{L}_{\psi}^{n}\left(\frac{\prod_{k=0}^{n-1}\psi \circ T^k \circ \tau^{-1}}{\prod_{k=0}^{n-1}\psi \circ T^k}f\right)\right)(x).
\end{aligned}$$

Since $\mathcal{L}_{\psi}^{*}\nu = \nu$, one obtains that

$$\int f \circ \tau d\nu = \int \frac{\prod_{k=0}^{n-1}\psi \circ T^k \circ \tau^{-1}}{\prod_{k=0}^{n-1}\psi \circ T^k}f\, d\nu.$$

Moreover, since $f$ is arbitrary, we conclude that $\tau_*\nu|_{\tau(U)} \ll \nu|_{\tau(U)}$, and

$$\frac{d\tau_*\nu|_{\tau(U)}}{d\nu|_\tau(U)} = \frac{\prod_{k=0}^{n-1}\psi \circ T^k \circ \tau^{-1}}{\prod_{k=0}^{n-1}\psi \circ T^k} = \frac{\prod_{k=0}^{\infty}\psi \circ T^k \circ \tau^{-1}}{\prod_{k=0}^{\infty}\psi \circ T^k},$$

since $T^k\tau^{-1}(x) = T^k(x)$ for all $k \geq n$. □

*Remark.* The above theorem immediately checks the requirements in the definition 7.5 of weakly Gibbs states: it establishes the absolute continuity of $\tau_*\mu$ with respect to $\mu$ and gives a corresponding family of multipliers

$$R_\tau = \frac{\prod_{k=0}^{\infty}\psi\big(T^k \circ \tau^{-1}(x)\big)}{\prod_{k=0}^{\infty}\psi\big(T^k(x)\big)}. \tag{7.10}$$

Thus our problem consists in establishing the properties of the product in the right hand side of (7.10).

**Theorem 7.7.** *Let $T$ be a Manneville-Pomeau type map. Let $\mu$ be the absolutely continuous invariant measure for $T$. Then $\mu$ is not a Gibbs, but is a weakly Gibbsian measure: the multipliers $R_\tau$, given by (7.7), are well-defined non-negative integrable functions, but not all of them are positive and continuous.*

*Proof.* Let $h$ be the density of $\mu$ and let us introduce a normalized transfer operator $\mathcal{L}_0$ corresponding to $\psi_0$, which is given by

$$\psi_0(x) = \begin{cases} \dfrac{h(x)}{h(T(x))} \dfrac{1}{|T'(x)|} & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases} \tag{7.11}$$

Hu in [13] showed that $\psi_0$ is a continuous function on $X$ satisfying $0 \le \psi_0(x) \le 1$, $\psi_0(x) = 1$ if and only if $x = 0$, and the only zero of $\psi_0$ at $p_+$, which was obtained by doubling the point $p$. Note that $T(p_+) = 0$.

Consider also the transfer operator $\mathcal{L}$ corresponding to $\psi = \dfrac{1}{|T'|}$. It is well known [18] that the transfer operator $\mathcal{L}$ maps $L^1(m)$ to itself ($m$ denotes the Lebesgue measure), the density $h = \dfrac{d\mu}{dm}$ satisfies $\mathcal{L}h = h$, and

$$\int \mathcal{L}f \, dm = \int f \, dm \tag{7.12}$$

for all $f \in L^1(m)$ and thus, $\mathcal{L}^* m = m$.

The normalized transfer operator $\mathcal{L}_0$ has the following properties:

1) $\mathcal{L}_0 \mathbb{I} = \mathbb{I}$, where $\mathbb{I}(x) = 1$ for all $x$.

2) operators $\mathcal{L}$ and $\mathcal{L}_0$ are related by the following formula

$$\mathcal{L}_0 f = \frac{1}{h} \mathcal{L}(hf) \quad \text{for all } f.$$

3) for every $f \in L^1(\mu)$

$$\int \mathcal{L}_0 f \, d\mu = \int f \, d\mu,$$

and hence $\mathcal{L}_0^* \mu = \mu$.

The last property of $\mathcal{L}_0$ follows easily from the corresponding property of $\mathcal{L}$. Indeed,

$$\int \mathcal{L}_0(f) d\mu = \int \frac{1}{h} \mathcal{L}(hf) h dm = \int \mathcal{L}(hf) dm = \int hf d\mathcal{L}^* m = \int f h dm = \int f d\mu.$$

Therefore we can apply Theorem 7.6 to $\mu$: substitute the expression (7.11) for $\psi_0$ into corresponding expression (7.9) for $R_\tau$. Let $\tau \in \mathcal{E}_n$ and assume that none of the points $\{T^k \circ \tau^i(x) |\ k = 0, \dots, n-1,\ i = -1, 0\}$ is equal to 0. Then, taking into account that $T^n \tau^{-1}(x) = T^n(x)$, we obtain

$$R_\tau(x) = \frac{h(\tau^{-1}(x))}{h(x)} \frac{|(T^n)'(x)|}{|(T^n)'(\tau^{-1}(x))|}.$$

The part of the previous formula involving the derivative of $T^n$ depends continuously on $x$ and is positive. The ratio $h(\tau^{-1}(x))/h(x)$ can be arbitrary large (small). Indeed, suppose $\tau \in \mathcal{E}_n$ and $\tau : \bar{I}_{0,\dots,0} \to I_{i_0,\dots,i_{n-1}}$, where $i_0 = 1$. The density $h(x)$ is bounded on $I_{i_0,\dots,i_{n-1}}$, on the other hand, since $h(t)$ is singular at $t = 0$, $R_\tau$ is singular as well.

It is also clear, that any other family of multipliers $\{\tilde{R}_\tau\}$, satisfying (7.7) of definition 7.5, can differ from the obtained densities only on a set of measure 0. Thus they can not be positive and continuous. Hence $\mu$ is not a Gibbs state, but is weakly Gibbsian. $\square$

In the following section we study the absolutely continuous invariant measure $\mu$ in the symbolic representation. We prove that there exists an almost surely absolutely convergent potential $U$ such that the conditional probabilities of $\mu$ can be represented in a standard Gibbsian way. This strengthens the result of theorem 7.7 and establishes a relation between the notions of weakly Gibbsian measures, commonly used in statistical mechanics, and the definition 7.5. Moreover, certain properties of the measure $\mu$ can be understood from the decay of the potential $U$.

### Symbolic dynamics: the potential

Consider again the coding $\pi : \{0,1\}^{\mathbb{Z}_+} \to X$. The question we want to deal with here is to see which kind of a potential (in the sense of equilibrium statistical mechanics) is associated to $\nu = \pi^*\mu$ and how the properties of this potential can be related to the decay of correlations.

Let us introduce some notation: put $\Omega = \{0,1\}^{\mathbb{Z}_+} = \{\omega = (\omega_i) : \ \omega_i \in \{0,1\}, i \in \mathbb{Z}_+\}$, $[\omega_0, \dots, \omega_{n-1}]$ is the cylinder with first coordinates $\omega_0, \dots, \omega_{n-1}$. If $\omega \in \Omega$ and $\Lambda \subseteq \mathbb{Z}_+$ then $\omega_\Lambda$ is a projection of $\omega$ to $\{0,1\}^\Lambda$, so $\omega_{\{i\}} = \omega_i$. For $\Lambda, \Lambda' \subseteq \mathbb{Z}_+$, $\Lambda \cap \Lambda' = \varnothing$, and $\xi \in \{0,1\}^\Lambda$, $\eta \in \{0,1\}^{\Lambda'}$, we let $\zeta = \xi\eta \in \{0,1\}^{\Lambda \cup \Lambda'}$ be such that $\zeta|_\Lambda = \xi$ and $\zeta|_{\Lambda'} = \eta$. For any $\Lambda \subseteq \mathbb{Z}_+$ denote by $\Lambda^c$ the complement of $\Lambda$ in $\mathbb{Z}_+$. For any $\xi \in \{0,1\}^\Lambda$, $\eta \in \{0,1\}^{\Lambda^c}$, the conditional probability of observing $\xi$ on $\Lambda$ given $\eta$ on the complement will be denoted by $\nu\left(\omega|_\Lambda = \xi \,\middle|\, \omega|_{\Lambda^c} = \eta\right)$, or, shortly $\nu(\xi|\eta)$. Finally, $\bar{0}$ and $\bar{1}$ are the configurations consisting entirely of 0's and 1's.

We start by observing that $\nu$ is certainly not Gibbsian in the usual sense. The reason is that the $\nu$-probability of the cylinder $\{\omega : \omega_0 = \omega_1 = \ldots = \omega_n = 0\}$ only decays polynomially, see (7.30). As a result the relative entropy density $i(\delta_{\bar{0}}|\nu)$ between the Dirac measure on the configuration of all zeros $\delta_{\bar{0}}$ and $\nu$ vanishes:

$$i(\delta_{\bar{0}}|\nu) = \lim_{n\to\infty} \frac{1}{n}\, I_n(\delta_{\bar{0}}|\nu) = 0,$$

where

$$I_n(\lambda|\rho) = I(\lambda_n|\rho_n) = \int \log \frac{d\lambda_n}{d\rho_n}\, d\lambda_n = \int \frac{d\lambda_n}{d\rho_n} \log \frac{d\lambda_n}{d\rho_n}\, d\rho_n$$

is the Kullback-Liebler information between the projections $\lambda_n$, $\rho_n$ of the measures $\lambda, \rho$ onto cylinders of length $n+1$, i.e., $\Omega_n = \{0,1\}^{n+1}$. But by the variational principle of statistical mechanics, see e.g. [36], this means that $\nu$ cannot be Gibbsian (since then $\delta_{\bar{0}}$ would be Gibbsian with the same potential which is absurd).

Using (7.7), the conditional probabilities for $\nu$ can be written [16, 29] as

$$\nu(\omega_0, \ldots, \omega_n | \omega_{n+1}, \omega_{n+2}, \ldots) = \frac{R_{\tau_{[\omega_0,\ldots,\omega_n]}}\big(\pi^{-1}(\omega_0 \ldots \omega_n \omega_{n+1} \ldots)\big)}{\sum\limits_{\omega'_0,\ldots,\omega'_n} R_{\tau_{[\omega'_0,\ldots,\omega'_n]}}\big(\pi^{-1}(\omega'_0 \ldots \omega'_n \omega_{n+1} \ldots)\big)}, \tag{7.13}$$

where $\tau_{[\omega_0,\ldots,\omega_n]} \in \mathcal{E}_{n+1}$ is a conjugating homeomorphism, mapping $\bar{I}_{\omega_0,\ldots,\omega_n}$ to $\bar{I}_{1,\ldots,1}$. Note that we have chosen here $\bar{1}$ as a reference state. It is easy to see that actually (7.13) does not depend on the choice of the reference state.

In particular, for $n = 0$ we get that the conditional probability to find $\omega_0 \in \{0,1\}$ at the origin while the rest of the configuration on $\{1,2,\ldots\}$ is $\omega_{0^c} = (\omega_1, \omega_2, \ldots)$, is simply given by

$$\nu(\omega_0 | \omega_1, \omega_2, \ldots) = \frac{\psi_0(\pi^{-1}(\omega))}{\psi_0(\pi^{-1}(\omega^0)) + \psi_0(\pi^{-1}(\omega))}, \tag{7.14}$$

where $\omega = (\omega_0, \omega_1, \ldots)$ and $\omega^0$ is $\omega$ "flipped" at the origin, i.e., $\omega^0_0 = 1 - \omega_0$ and $\omega^0_i = \omega_i$ for $i \neq 0$. Since $\psi_0$ is continuous, we immediately conclude that the non-Gibbsian character of $\nu$ is not related to the presence of essential discontinuities in the conditional probabilities (as in the case of the restricted Ising model of the introduction). However, $\nu(\omega_0|\omega_{0^c})$ is not uniformly non-null: it is easily seen that for the continuous version of the conditional probabilities one has

$$\nu(\omega_0 = 1 | \omega_{0^c} = \bar{0}) = 0 \quad \text{and} \quad \nu(\omega_0 = 0 | \omega_{0^c} = \bar{0}) = 1,$$

where $\bar{0}$ denotes the configuration of all zeros. Therefore we expect to find a potential $U(\Lambda, \omega)$ for which the sums that form the local Hamiltonian

$$H^U_\Lambda(\omega) = \sum_{A \cap \Lambda \neq \varnothing} U(A, \omega) \tag{7.15}$$

will diverge at $\omega = \bar{0}$. More precisely we have the following:

**Theorem 7.8.** *There exists a translation invariant potential $U(\Lambda, \omega)$ with the following properties*

*1) $U(\Lambda, \omega) = 0$ unless $\Lambda = [i, j]$,* *2)* $\exists \delta > 0$, $\exists l : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ *such that on the set* $K := \{\omega : l(\omega) < \infty\}$ *we have the estimate*

$$|U([0,n],\omega)| \leq \begin{cases} C_2(\omega), & n < l(\omega), \\ C_1(\omega)\exp(-\delta n), & n \geq l(\omega), \end{cases}$$

*for some* $C_i(\omega) < \infty, \omega \in K$*.*

*3)* $\nu$ *is weakly Gibbsian with potential* $U$*, see (7.29).*

*Proof.* We consider the Kozlov potential (see [17]) with reference state $\omega = \bar{1}$:

$$U([i,j],\omega) = \log \frac{\nu(1_{\{i\}}\omega_{(i,j]}|1_{[i,j]^c})\nu(\omega_{[i,j)}1_{\{j\}}|1_{[i,j]^c})}{\nu(\omega_{[i,j]}|1_{[i,j]^c})\nu(1_{\{i\}}\omega_{(i,j)}1_{\{j\}}|1_{[i,j]^c})}. \tag{7.16}$$

From the inequality $|\log a - \log b| \leq |a-b|/\min\{a,b\}$, we have the estimate

$$|U([0,n],\omega)| \leq c(\omega)\varphi_n(\omega), \tag{7.17}$$

where

$$c(\omega)^{-1} = \frac{1}{2}\min_n \min_{a,b} \nu(a|\omega_{(0,n-1)}b_n 1_{(n,\infty)}) \geq 0 \tag{7.18}$$

with equality only for $\omega = \bar{0}$, and where

$$\varphi_n(\omega) = \sup_{\xi,\xi'} \left|\nu(\omega_0|\omega_{[1,n-1]}\xi_{[n,\infty)}) - \nu(\omega_0|\omega_{[1,n-1]}\xi'_{[n,\infty)})\right|. \tag{7.19}$$

Using expression (7.14) for the conditional probabilities, we obtain

$$\begin{aligned} \varphi_n(\omega) &= \sup_{\xi,\xi'} \left| \frac{\psi_0 \circ \pi(\omega_{[0,n-1]}\xi_{[n,\infty)})}{\psi_0 \circ \pi(\omega^0_{[0,n-1]}\xi_{[n,\infty)}) + \psi_0 \circ \pi(\omega_{[0,n-1]}\xi_{[n,\infty)})} \right. \\ &\qquad \left. - \frac{\psi_0 \circ \pi(\omega_{[0,n-1]}\xi'_{[n,\infty)})}{\psi_0 \circ \pi(\omega^0_{[0,n-1]}\xi'_{[n,\infty)}) + \psi_0 \circ \pi(\omega_{[0,n-1]}\xi'_{[n,\infty)})} \right| \\ &\leq C_3 \left[ \sup_{x,y \in I_{\omega_0\omega_1\cdots\omega_{n-1}}} |\psi_0(x) - \psi(y)| \right. \\ &\qquad \left. + \sup_{x,y \in I_{\omega_0^0\omega_1\cdots\omega_{n-1}}} |\psi_0(x) - \psi_0(y)| \right] \end{aligned} \tag{7.20}$$

with $C_3 = 2\sup_\xi \frac{1}{\psi_0\circ\pi(\xi) + \psi_0\circ\pi(\xi^0)} < \infty$. Estimating (7.20) is only problematic in the case where $x$ or $y$ are very close to zero. Since $h$ is bounded and Lipschitz on $(\epsilon, 1]$ for any $\epsilon > 0$ (see [13]), we know that if $\epsilon < x \leq y$ for some $\epsilon > 0$ then there exists a constant $C_4 = C_4(\epsilon)$ such that

$$|\psi_0(x) - \psi_0(y)| \leq C_4|x-y|. \tag{7.21}$$

Of course, things remain bad for $x = 0$ and we must therefore restrict ourselves to 'good' configurations. We first define what we mean by this. Put $\beta = \int \nu(d\omega)\omega_0 = \mu(I_1) > 0$ and define

$$\ell(\omega) = \inf\left\{n \in \mathbb{N}_0 : \frac{1}{k}\sum_{i=0}^{k-1}\omega_i > \frac{1}{2}\beta \text{ for all } \; k \geq n \; \right\}. \tag{7.22}$$

We say that $\omega$ is 'good' if $\ell(\omega) < \infty$ and we collect them in the set

$$K = \{\omega : \ell(\omega) < \infty\}. \tag{7.23}$$

Note that $K$ is a set in the tail field. Indeed, if two configurations $\omega$ and $\omega'$ are such that

$$\left\{i \in \mathbb{N}_0 : \; \omega_i \neq \omega_i'\right\}$$

is a finite set then $\ell(\omega) < \infty$ if and only if $\ell(\omega') < \infty$. Thus if $\omega \in K$ then $\omega^0 \in K$ as well. For $\omega \in K$, we define

$$\epsilon = \epsilon(\omega) = \frac{1}{2}\inf\left\{x \in I_{\omega_0\ldots\omega_{\ell(\omega)-1}} \cup I_{\omega_0^0\ldots\omega_{\ell(\omega^0)-1}}\right\}. \tag{7.24}$$

By the definition of $\ell(\omega)$ and $K$, $\epsilon(\omega) > 0$ for every $\omega \in K$. Moreover, for every $n \geq \ell(\omega)$, if $x \in I_{\omega_0\ldots\omega_{n-1}} \cup I_{\omega_0^0\ldots\omega_{n-1}}$ then $x > \epsilon(\omega)$.

Combining (7.20) and (7.21), we have that

$$\varphi_n(\omega) \leq C(\omega)\left(|I_{\omega_0,\ldots,\omega_{n-1}}| + |I_{\omega_0^0\ldots\omega_{n-1}}|\right). \tag{7.25}$$

Now use that $1/|T'(x)| \leq \mathrm{e}^{-\delta'}$ for $x \in [p, 1]$ and $\delta' = \log\lambda > 0$; this gives the estimate

$$|I_{\omega_0\omega_1\ldots\omega_{n-1}}| \leq \exp(-\delta'\sum_{i=0}^{n-1}\omega_i), \tag{7.26}$$

where we used that $|I_{\omega_0\omega_1\ldots\omega_{n-1}}| = |(T^n)'(\bar{x})|^{-1}$ for some $\bar{x} \in I_{\omega_0\omega_1\ldots\omega_{n-1}}$. Therefore for $\omega \in K$ and $n \geq \ell(\omega)$ we have

$$\varphi_n(\omega) \leq C(\omega)\mathrm{e}^{-\delta n}, \tag{7.27}$$

with $\delta = \frac{1}{2}\beta\delta'$. For $\omega \in K$ and $n \leq l(\omega)$ we have the trivial bound

$$\varphi_n(\omega) \leq 2. \tag{7.28}$$

Together with (7.16)-(7.18), this finishes the proof of claims 1 and 2 of the theorem and shows that the potential is absolutely convergent on the set $K$.

In order to prove that $\nu$ is weakly Gibbsian with potential $U$ we still have to establish two facts:

1) the potential $U$ is absolutely convergent on a set of $\nu$-measure one,

2) $\nu$ is consistent with the potential, i.e.,

$$\nu(\omega_0|\omega) = \frac{\exp\left(-H_{\{0\}}(\omega)\right)}{\exp(-H_{\{0\}}(\omega)) + \exp(-H_{\{0\}}(\omega^0))}, \quad \nu-\text{a.s.}, \tag{7.29}$$

where $H_{\{0\}}$ is the local Hamiltonian defined in (7.15).

The second point follows from the first one and from the continuity of the conditional probabilities (see e.g. [22, 23]). The first fact is a simple consequence of the ergodic theorem:

$$\nu(K^c) \le \nu\Big(\Big\{\omega : \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \omega_i \le \frac{\beta}{2}\Big\}\Big) = 0.$$

□

Theorem 7.8 states that the potential $U([0,n],\omega)$ decays exponentially for $n$ larger than some configuration dependent "correlation length" $\ell(\omega)$. As we have seen above, the correlations for the MP maps decay polynomially: $\rho_n = O(n^{-1/\alpha+1})$, where $\alpha \in (0,1)$ is the parameter of the MP map. It turns out that the "distribution" of the correlation length $\ell(\omega)$ is closely related to the above decay of correlations. This is the content of the following proposition:

**Theorem 7.9.** *One has the following estimates of $\nu(\{\omega : \ell(\omega) \ge n\})$ depending on the parameter $\alpha$:*

*i) for $\alpha \in (1/2, 1)$ there exist constants $C_1, C_2 > 0$ such that*

$$C_1 n^{-\frac{1}{\alpha}+1} \le \nu(\{\omega : \ell(\omega) \ge n\}) \le C_2 n^{-\frac{1}{\alpha}+1};$$

*ii) for $\alpha = 1/2$ there exist constants $C_1, C_2 > 0$ such that we have*

$$C_1 n^{-1} \le \nu(\{\omega : \ell(\omega) \ge n\}) \le C_2 n^{-1}\log n;$$

*iii) for $\alpha \in (0, 1/2)$ and any $\delta > 0$ there exist constants $C_1, C_2 > 0$ such that*

$$C_1 n^{-\frac{1}{\alpha}+1} \le \nu(\{\omega : \ell(\omega) \ge n\}) \le C_2 n^{-\frac{1}{\alpha}+1+\delta}.$$

We are going to use the following result in the proof of the above statement.

**Theorem 7.10.** *Let $\Sigma = A^{\mathbb{Z}_+}$, where $A = \{1, \dots, N\}$, and let $\nu$ be a shift-invariant probability measure on $\Sigma$. For a bounded function $f : A \to \mathbb{R}$ with $\int f\, d\nu = 0$, denote*

$$\rho_k(f) = \Big|\int f(\omega_0) f(\omega_k)\, d\nu\Big|, \quad k \in \mathbb{N}.$$

*Suppose that for some $m \ge 1$*

$$\zeta_m := \sum_{k=1}^{\infty} \rho_k^{1/m} < \infty.$$

*Then there exists a constant $C > 0$ such that for all $n \geq 1$ one has*

$$\int \Big| \sum_{k=0}^{n-1} f(\omega_k) \Big|^{2m} d\nu \leq C \big(\zeta_m\big)^m n^m.$$

*Proof.* The cases $m = 1$ and $m = 2$ are in fact Lemmas 2 and 4 in [2, Chap.4,p.172]. The statement can be easily generalized for any other integer $m > 2$. The remaining case $m = k + \delta$, where $k$ is an integer and $\delta \in (0, 1)$ can be proved along the lines of Lemma 7.4 [8, p.225]. One has to stress that Lemma 7.4 in [8] is proved under very different assumptions: discrete Markov chains with exponential mixing; its proof, however, can be adopted to our purposes with minor modifications. $\square$

**Proof of Theorem 7.9:** The lower bound is easy:

$$\nu(\{\omega : \ell(\omega) \geq n\}) \geq \mu(I_{00\ldots0}),$$

where the number of zeros is $n$. Since $h(x) \geq C_3 x^{-\alpha}$ and $m(I_{00\ldots0}) \geq C_4 n^{-1/\alpha}$:

$$\mu(I_{00\ldots0}) \geq \int_0^{C_4 n^{-1/\alpha}} C_3 x^{-\alpha} dx = C_1 n^{1-\frac{1}{\alpha}}. \tag{7.30}$$

The upper bound is more difficult: Put $S_n(\omega) := \sum_{i=0}^{n-1}(\omega_i - \beta)$, where we recall that $\beta = \int \omega_0 d\nu > 0$.

We have to estimate $\nu(\{\omega : \ell(\omega) \geq n\})$, i.e.,

$$\nu\left(\left\{\omega : \exists k \geq n \ \text{ such that } \left|\frac{S_k}{k}\right| > \frac{\beta}{2}\right\}\right) = \nu\left(\left\{\omega : \sup_{k\geq n} \left|\frac{S_k}{k}\right| > \frac{\beta}{2}\right\}\right). \tag{7.31}$$

According to Theorem 12 in [15] the following conditions are equivalent:

1) $\nu\left(\left\{\omega : \sup_{k\geq n} \left|\frac{S_k}{k}\right| > \frac{\beta}{2}\right\}\right) = O(n^{-\gamma})$ as $n \to \infty$;

2) $\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \frac{\beta}{2}\right\}\right) = O(n^{-\gamma})$ as $n \to \infty$.

Therefore these two probabilities have a similar asymptotic behavior. The second quantity, however, is much easier to deal with.

Let us start with the case $\alpha \in [1/2, 1)$. By the Chebyshev inequality

$$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \frac{\beta}{2}\right\}\right) \leq \frac{4\int |S_n|^2 \, d\nu}{n^2\beta^2} \leq C\frac{1}{n}\sum_{k=0}^{n-1} \rho_k.$$

Taking into account that $\rho_k = O(k^{-1/\alpha+1})$ for $k \geq 1$, we conclude that for some $C_2$

$$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \frac{\beta}{2}\right\}\right) \leq C_2 n^{-1/\alpha+1} \quad \text{for } \alpha \in (1/2, 1),$$

and

$$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \frac{\beta}{2}\right\}\right) \le C_2 n^{-1}\log n \quad \text{for } \alpha = 1/2.$$

The above argument can not produce an estimate decaying faster than $1/n$. Therefore, for $\alpha \in (0, 1/2)$ we have to use higher moments of $S_n$ in order to obtain better estimates.

Consider $\alpha \in (0, 1/2)$ and take any sufficiently small $\delta \in (0,1)$ such that $m = \frac{1-\alpha}{\alpha}(1-\delta) \ge 1$. Since $\rho_k = O(k^{-1/\alpha+1})$ one has

$$\zeta_m = \sum_{k=1}^{\infty} \rho_k^{1/m} \le C \sum_{k=1}^{\infty} k^{-1/(1-\delta)} < +\infty.$$

Using the Chebyshev inequality and the estimate from Theorem 7.10 we conclude that there exists a constant $C_2$ such that

$$\nu\left(\left\{\omega : \left|\frac{S_n}{n}\right| > \frac{\beta}{2}\right\}\right) \le C_2 n^{-m} = C_2 n^{-\frac{1}{\alpha}+1+\delta'},$$

where $\delta' = (1/\alpha - 1)\delta$. This finishes the proof of the upper bound. $\square$

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# Chapter 8

# Numerical Estimation of the Correlation Entropies

In the previous chapters we have seen that the family of correlation entropies contains a substantial amount of information about the underlying dynamics. For example, under very mild assumptions a Legendre transform of the correlation entropies gives an upper estimate of the multifractal spectrum of local entropies; non-uniformly hyperbolic behavior of a dynamical system (like those of a Manneville-Pomeau map) can be detected by the correlation entropies as well. Hence it is important that the correlation entropies can be estimated consistently. We consider this estimation problem in particular for situations where no equations of motion (i.e., maps or differential equations) are known, but where a finite and sufficiently long segment of a typical orbit (or a corresponding time series) is known. We discuss methods for estimation, and test them on some model examples: the skew-tent map, the Hénon map and the Manneville-Pomeau map. We had the following motivation for the choice of these dynamical systems. For the tent map one can actually compute the correlation entropies explicitly, and hence we can evaluate the consistency, quality and the speed of convergence of our estimators. For the second example — the Hénon map, the explicit expressions for the entropies are not known. However some estimates have been given for the correlation entropy of order 2 [2] and for the topological entropy [9] which we believe to be equal to the correlation entropy of order 0. We would like to remind that the Hénon family is not hyperbolic, but for a large set of parameters some properties, similar to those of hyperbolic systems, have been established. The third example — the family of Manneville-Pomeau maps, was originally suggested as a simple model of intermittent behavior. We have seen earlier that this results in a *phase transition* in the family of correlation entropies: $H(q) > c > 0$ for $q \leq 1$, and $H(q) \equiv 0$ for $q > 1$. We will see that our methods of numerical estimation of the correlation entropies are capable of detecting this phase transition.

**Setup.** We consider the following model for deterministic time series. Suppose that a dynamical is given by some transformation $f : X \to X$. An orbit is then a sequence $\{x_i\}$ such that

$$x_{i+1} = f(x_i) \quad \text{for all} \quad i.$$

For an observable $\varphi : X \to \mathbb{R}$, the *time series* , corresponding to an orbit $\{x_i\}$, is given by

$$y_i = \varphi(x_i) = \varphi \circ f^i(x_0). \tag{8.1}$$

h The idea is that we record not the complete state of the system $x_i$, but only the values of the observable, namely, $y_i = \varphi(x_i)$. Under certain generic conditions, the so-called *Reconstruction Theorem* [10] states that some characteristics of the original dynamics such as dimension or entropy of the $\omega$-limit set or attractor can be recovered from a time series (8.1).

In the subsequent examples, the function $\varphi$ will be the identity or 'almost' the identity. Therefore we keep the same notation $\{x_i\}$ for the orbit and for the corresponding time series.

In this chapter we treat 3 examples. The first two – the skew-tent map and the Manneville-Pomeau map, are interval maps. In these cases $\varphi$ will be equal to the identity, i.e., the orbit and the corresponding time series will be the same. The last example – the Hénon map, is a transformation of $\mathbb{R}^2$, given by

$$\begin{aligned} x_{i+1} &= 1 - ax_i^2 + y_i, \\ y_{i+1} &= bx_i. \end{aligned}$$

In this case, $\varphi$ will be a projection on the first coordinate, i.e. the time series is $\{x_i\}_{i\geq 0}$. In the sequel, the time series, for which we estimate the correlation entropies, will be denoted by $\{x_i\}$.

From now on we assume that we have at our disposal a finite time series $\{x_i\}_{i=1}^N$, where $x_i \in \mathbb{R}$ and $N$ is assumed to be large.

The correlation entropies are defined for invariant measures. In the case of a finite series, the empirical probability measure $\mu_N$ given by

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i},$$

where $\delta_y$ is the Dirac measure at the point $y$. Our assumption will be that the time series comes from a dynamical system, admitting an SRB or a physical measure $\mu$, and that $\{x_i\}$ is a generic trajectory for $\mu$. Hence we may assume that for large $N$, the empirical measure $\mu_N$, defined above, is close (in the weak topology) to $\mu$.

## 8.1 Direct approach.

**Step 1. Estimate of $\mu(\mathcal{B}_n(x,\varepsilon))$.** In order to estimate the correlation entropies, we first have to provide consistent estimates of $\mu(\mathcal{B}_n(x_i,\varepsilon))$. We use the obvious estimate: for $i = 1, \dots, N-n$

$$C(x_i,\varepsilon,n) = \frac{1}{N-n-1}\#\{j \neq i : d_n(x_i,x_j) < \varepsilon\}.$$

Assuming ergodicity, we get

$$C(x_i, \varepsilon, n) \to \mu(\mathcal{B}_n(x_i, \varepsilon)) \text{ as } N \to \infty.$$

However for certain values of $i$, $\varepsilon$, and $n$ it is possible that $C(x_i, \varepsilon, n) = 0$, causing problems in the implementation of estimates of the generalized entropies of order $q < 1$.

**Step 2. Estimate of $I(q, \varepsilon, n)$.** Using the above estimate $C(x_i, \varepsilon, n)$, we let

$$\hat{I}(q, \varepsilon, n) = \frac{1}{N-n} \sum_{i=1}^{N-n}{}^{*} \big(C(x_i, \varepsilon, n)\big)^{q-1}.$$

where $\sum^*$ is the summation over all $i$ such that $C(x_i, \varepsilon, n) > 0$. Obviously, this is necessary only for $q < 1$. This introduces bias into our estimates: we have to leave out those points which have no neighbors of the required type, but these points should, due to the negative exponent, contribute with a high weight.

In order to estimate $\hat{I}(q, \varepsilon, n)$ one has to make $O(N^2)$ operations. This can be very time consuming even on a powerful workstation. The complexity $N^2$ comes from the fact that we have substituted two integrals by finite sums, each containing $N$ elements. One can substantially decrease the required time by performing the following modification of the above scheme. Given a time series $\{x_i\}_{i=1}^{N}$, choose some number of reference points $N_{ref}$ such that $N_{ref} \ll N$. Generate $N_{ref}$ randomly chosen indexes $i(k) \in \{1, \ldots, N\}$, $k = 1, \ldots, N_{ref}$. For each $i(k)$ estimate $\mu(\mathcal{B}_n(x_{i(k)}, \varepsilon))$ as described above, namely

$$C(x_{i(k)}, \varepsilon, n) = \frac{1}{N-n-1} \# \Big\{ j \neq i(k) : d_n(x_j, x_{i(k)}) < \varepsilon \Big\},$$

and let

$$\hat{I}(q, \varepsilon, n) = \frac{1}{N_{ref}} \sum_{k=1}^{N_{ref}}{}^{*} \big(C(x_{i(k)}, \varepsilon, n)\big)^{q-1}. \tag{8.2}$$

The complexity of this approach is $O(N_{ref} \times N)$. Numerical experiments show that a choice of $N_{ref}$ equal to a fraction of $N$ does not have a large effect on the value of $\hat{I}(q, n, \varepsilon)$, while the gain in the CPU time is substantial.

**Step 3. Estimates of $H_\mu(f, q)$.** For sufficiently small $\varepsilon > 0$ and sufficiently large $n$ one assumes that

$$I(q, \varepsilon, n) \approx C e^{-(q-1) H_\mu(f,q) n}, \tag{8.3}$$

where $H_\mu(f, q)$ is the generalized entropy of order $q$.

Figure 8.1 shows estimated values $\log \hat{I}(q, \varepsilon_k, n)$ in the case of the skew-tent map and for the parameter values $q = 2.0$, $\varepsilon_k = (2k)^{-2}$, $k = 1, \ldots, 20$, and $n = 1, \ldots, 20$. A linear decay of $\log \hat{I}(q, \varepsilon_k, n)$ with $n$ is clearly visible. We will use various estimates of the speed of this decay as estimators of the correlation

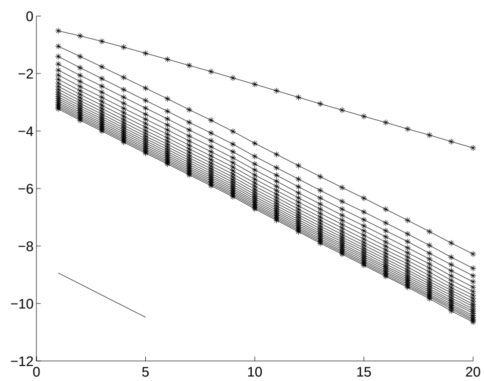



Figure 8.1: Graphs of $\log \hat{I}(2, \varepsilon_k, n)$ for $\varepsilon_k = (2k)^{-2}$, $k = 1, \dots, 20$, as functions of $n$. The slope of a short straight line in the lower left corner is equal to the theoretical value of the correlation entropy of order 2.

entropy $H_\mu(f, q)$. There are several ways to estimate the rate of decay. Let us discuss some of them.

We can estimate the rate of decay by computing the slope of a chord, connecting $\log \hat{I}(q, \varepsilon_k, n_1)/(q-1)$ and $\log \hat{I}(q, \varepsilon_k, n_2)/(q-1)$:

$$\hat{H}_\mu(f, q) = -\frac{1}{(q-1)(n_2 - n_1)} \log \frac{\hat{I}(q, \varepsilon, n_2)}{\hat{I}(q, \varepsilon, n_1)}, \quad n_2 > n_1 \geq 1.$$

From all possible choices of $n_1$, $n_2$ ($n_2 \geq n_1$) we only consider the following two:

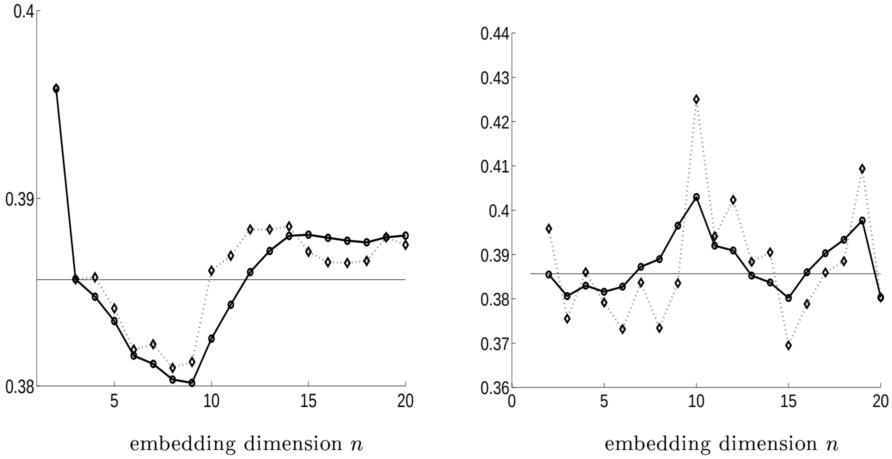
$$n_1 = 1,\, n_2 = n: \qquad \hat{H}_\mu^{(1)}(f, q, n) = -\frac{1}{(q-1)(n-1)} \log \frac{\hat{I}(q, \varepsilon, n)}{\hat{I}(q, \varepsilon, 1)}, \quad n > 1.$$

$$n_1 = n,\, n_2 = n+1: \qquad \hat{H}_\mu^{(2)}(f, q, n) = \frac{1}{(q-1)} \log \frac{\hat{I}(q, \varepsilon, n)}{\hat{I}(q, \varepsilon, n+1)}, \quad n > 1.$$

We will also consider the so-called Takens estimator

$$\hat{H}_\mu^{(3)}(f, q, n) = \frac{1}{q-1} \log \frac{\sum_{k=n}^{n_{max}-1} \hat{I}(q, \varepsilon, k)}{\sum_{k=n+1}^{n_{max}} \hat{I}(q, \varepsilon, k)},$$

where $n_{max}$ is the maximal embedding dimension for which $\hat{I}(q, \varepsilon, n)$ were computed. In the case of generalized dimensions, the Takens estimator [11] is known to have rather good properties, see [1].




(a) Estimates $\hat{H}_\mu^{(1)}(f,2,n)$ (dotted line) and $\hat{H}_\mu^{(4)}(f,2,n)$ (solid line).

(b) Estimates $\hat{H}_\mu^{(2)}(f,2,n)$ (dotted line) and $\hat{H}_\mu^{(3)}(f,2,n)$ (solid line).

Figure 8.2: Various estimators of the correlation entropy of order 2 for the skew-tent map with $p = 0.8$, $\varepsilon = 0.0025$.

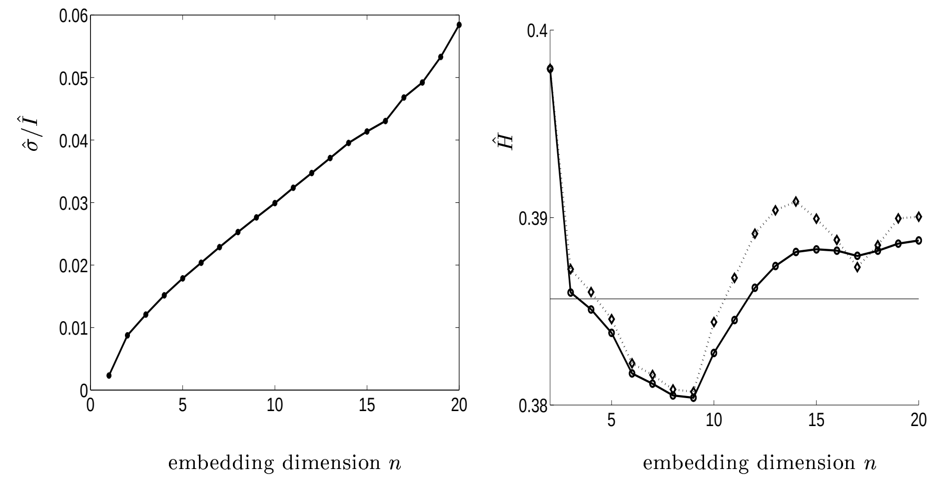
Finally, one could try to fit a straight line using the least squares method through the set of points $\{k, \log \hat{I}(q,\varepsilon,k)/(q-1)\}_{k=1}^{n}$, $n = 2, \ldots, n_{max}$, in the plane. The slope of this line, which we denote by $\hat{H}_\mu^{(4)}(f,q,n)$, will be our fourth estimator.

Let us try the above estimators for estimating the correlation entropy of order $q = 2.0$ of the skew-tent map, see chapter 1, with the parameter $p = 0.8$. The exact value of the correlation entropy of oder $q = 2.0$ is $-\log(p^2 + (1-p)^2) \approx 0.3857$. The other parameters are $N = 25.000$, $N_{ref} = 2.000$, and $n = 1, \ldots, 20$. In figure 8.1, the values of $-\log \hat{I}(2, \varepsilon_k, n)$ are plotted as functions of $n$ for $\varepsilon_k = (2k)^{-2}$, $k = 1, \ldots, 20$. Examining figure 8.1 it is clear that for the few first values of $k$ the values of $\varepsilon_k$ are not small enough. However, all larger values of $k$ produce lines with approximately (at least visually) correct slope. Figures 8.2.a and 8.2.b show the behavior of the estimators discussed above.

### 8.1.1 Estimating variance of $\hat{I}(2,\varepsilon,n)$.

In general, together with estimating $\hat{I}(q,\varepsilon,n)$, one also has to provide estimates of the variance. These estimates are necessary for further conclusions about the quality of estimates of the generalized entropies $\hat{H}_\mu(f,q)$.

Here we restrict ourselves to the simplest case $q = 2$. For $q = 2$ the expression




(a) Graph of $\hat{\sigma}(\hat{I}(2,\varepsilon,n))/\hat{I}(2,\varepsilon,n)$ as a function of the embedding dimension $n$, $n = 1,\ldots,20$, in the case of the skew-tent map with $p = 0.8$ and $\varepsilon = 0.0025$.

(b) Estimates of the correlation entropy of order 2: least squares with equal weights (dotted line) and with weights given by (8.5) (solid line).

Figure 8.3: Standard deviation $\hat{\sigma}(\log \hat{I}(2,\varepsilon,n))$ and the least square estimates with and without weights.

for $\hat{I}(2,\varepsilon,n)$ can be written in the following form

$$\hat{I}(2,\varepsilon,n) = \frac{1}{N_{ref}} \sum_{k=1}^{N_{ref}} W_k,$$

where

$$W_k = \frac{1}{N-1} \sum_{j \neq i(k)} Y_{i(k),j},$$

and $Y_{i(k),j}$ is either 0 or 1, depending on whenever the distance $d_n(x_j, x_{i(k)})$ is larger or smaller than $\varepsilon$. Since the indexes of the reference points were chosen randomly and independently, the quantities $\{W_k\}_{k=1}^{N_{ref}}$ are, to a good approximation, independent and identically distributed. Hence,

$$\mathrm{var}\big(\hat{I}(2,\varepsilon,n)\big) = \frac{1}{N_{ref}} \mathrm{var}\big(W_1\big).$$

The estimate of $\mathrm{var}\big(W_1\big)$ is well known to be

$$\hat{\sigma}^2(W_1) = \frac{1}{N_{ref}-1} \sum_{k=1}^{N_{ref}} \big(W_k - \overline{W}\big)^2, \tag{8.4}$$

where $\overline{W} = \sum_{k=1}^{N_{ref}} W_k/N_{ref}$ is the empirical mean, and hence

$$\hat{\sigma}(\hat{I}(2,\varepsilon,n)) = \frac{1}{\sqrt{N_{ref}}}\hat{\sigma}(W_1).$$

One has to stress that in (8.4) we estimate the variance of $W_1$ inside a given realization, and we disregard the dependence between $W_k$'s caused by the fact that all $W_k$'s are computed using the same time-series $\{x_i\}$.

From figure 8.3.a we can see that in the case of the skew-tent map with the parameters as in figure 8.1, the ratio $\hat{\sigma}(\hat{I}(2,\varepsilon,n))/\hat{I}(2,\varepsilon,n)$ is rather small: it does not exceed 6 percent. Using the fact that $\log(1+t) \approx t$ for $t$ with small $|t|$, we can conclude that

$$\log\big(\hat{I}(2,\varepsilon,n) \pm \hat{\sigma}(\hat{I}(2,\varepsilon,n))\big) \approx \log\hat{I}(2,\varepsilon,n) \pm \frac{\hat{\sigma}(\hat{I}(2,\varepsilon,n))}{\hat{I}(2,\varepsilon,n)}.$$

One could also use the obtained expressions for the standard deviation of $\log\hat{I}(2,\varepsilon,n)$ as inverse weights in the weighted least squares fitting. Namely, for every $n \geq 2$ minimize (in $a$ and $b$) the following function

$$\mathcal{L}(a,b,n) = \sum_{k=1}^{n} \frac{1}{s_k}\Big(\log\hat{I}(2,\varepsilon,k) - ak - b\Big)^2,$$

where

$$s_k = \frac{\hat{\sigma}(\hat{I}(2,\varepsilon,k))}{\hat{I}(2,\varepsilon,k)}, \tag{8.5}$$

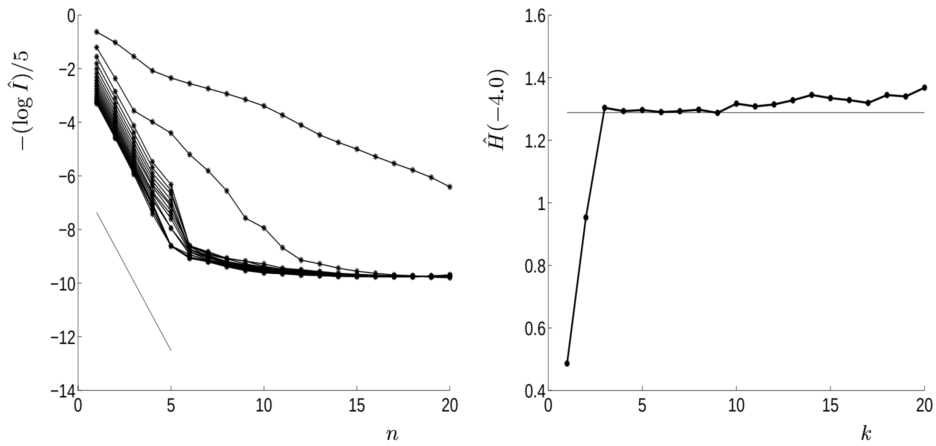
i.e., points with smaller standard deviation get bigger weights. The estimator of the correlation entropy then is minus the slope of the line, i.e, $-a$. One has to mention that since the "random" weights $s_k$ are observed quantities, the least square estimator based on these weights in principal can be biased.

In figure 8.3.b we compare the estimates obtained by least squares method with weights $s_k$, given by (8.5), and with the estimates obtained by letting $s_k \equiv 1$ (equal weights).

### 8.1.2 Estimation of the correlation entropy of order $q < 1$

Estimation of the correlation entropies becomes problematic if $q < 1$, and especially, if $q < 0$. In order to obtain sufficient information about the correlation entropies we do not have to consider negative $q$ with very large absolute values. For example, for the skew-tent map with a parameter $p = 0.8$, knowing the correlation entropies of orders $q \in [-4, 4]$, allows us to recover more than 99 percent of the information contained in the multifractal spectrum for local entropies.

Graphs in figure 8.4 show the results of a computation for the case of the skew-tent map with $p = 0.8$, and $q = -4.0$. It is clear that the estimates of $-\log\hat{I}(-4,\varepsilon_k,n)/5$ for $n > 5$ should not be used for estimating the correlation




(a): Estimates $-\log \hat{I}(-4, \varepsilon_k, n)/5$ as functions of $n$ for $\varepsilon_k = (2k)^{-2}$, $k = 1, \dots, 20$. The straight line in the lower left corner indicates the correct slope.

(b): Least Squares estimates using only 4 first points from the previous graph $-\log \hat{I}(-4.0, \varepsilon_k, n)/5$, $n = 1, \dots, 4$, for each $\varepsilon_k = (2k)^{-2}$, $k = 1, \dots, 20$.

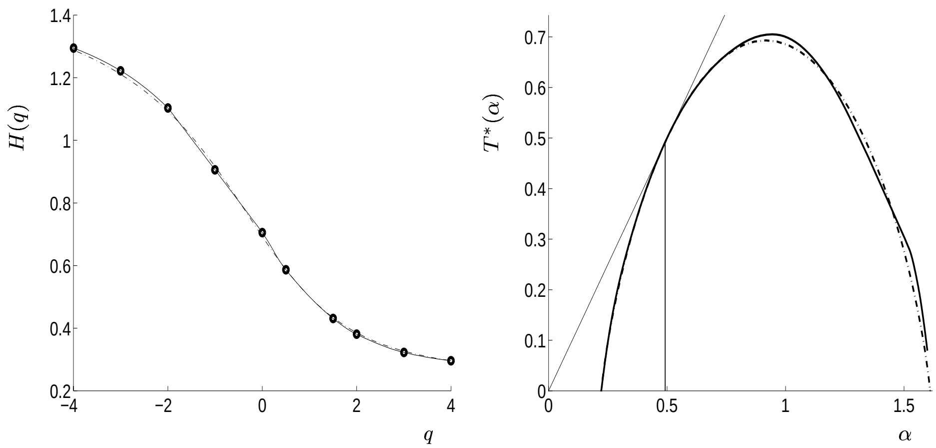
Figure 8.4: Estimation of the correlation entropy of order $q = -4.0$ in the case of the skew-tent map with parameter $p = 0.8$.

entropy. This could be explained using the formula for $\hat{I}(q, \varepsilon, n)$ in the case of negative $q$:

$$\hat{I}(q, \varepsilon, n) = \frac{1}{N_{ref}} \sum_{k=1}^{N_{ref}} \left( \frac{N-1}{\#\{j \neq i(k) : d(x_j, x_{i(k)}) < \varepsilon\}} \right)^{|q|+1},$$

where the sum is taken over all indexes $k$ with $C(x_{i(k)}, \varepsilon, n) := \{j \neq i(k) : d(x_j, x_{i(k)}) < \varepsilon\} > 0$. The largest contribution to the sum will come from the points $x_{i(k)}$ with a smallest $C(x_{i(k)}, \varepsilon, n)$. This is in fact what we want. On the other hand, the sample size $N(n)$, required for a reliable estimation of $1/C(x_i, \varepsilon, n)$, grows with the embedding dimension $n$. Starting from a certain embedding dimension $n$ (5 in our case), our sample size is not large enough to provide a reliable estimate of $1/C(x_{i(k)}, \varepsilon, n)$. By increasing the length of a time series $N$, we can increase the range of embedding dimensions $n$ (in our case it is $[1, 4]$), where linear scaling is evident, and is not hindered by the fluctuations of $1/C(x_{i(k)}, \varepsilon, n)$.

Similarly, choosing an extremely small $\varepsilon$, we get the same problem. It is evident from the graph 8.4.b, that for the chosen length of the time series $N = 25.000$, the values $\varepsilon_k = (2k)^{-2}$ for $k > 10$ are too small for a reliable estimate.




(a): Estimates of the generalized entropies of order $q = -4.0, \dots, 4.0$; circles represent the estimated values, solid line is obtained by interpolating with the quadratic splines. The almost invisible dotted line represents the true values of the generalized entropies.

(b): Estimate, on the basis of results in (a), of the spectrum of local entropies (solid line) and the true spectrum (dotted line). The point, where the straight line through the origin with a slope $45^o$ is tangent to the graph, gives an estimate of the measure-theoretic entropy $h_\mu$. In our case, $\hat{h}_\mu = 0.4918$ and the true value $h_\mu = 0.5004$.

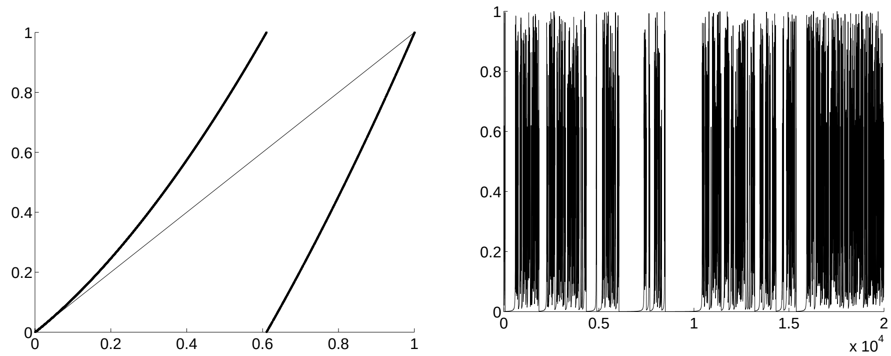
Figure 8.5: Generalized entropies $H(q)$ and the Legendre transform of $T(q) = (1-q)H(q)$ for the skew-tent map with $p = 0.8$.

## 8.2 Results

In this section we present estimates of the correlation entropies of different orders for three examples of dynamical systems. A motivation for the choice of these particular dynamical systems has been given in the introduction to this chapter.

### 8.2.1 Skew-tent map

Using methods discussed above, for the skew-tent map with $p = 0.8$ we obtained the results in figure 8.5.a. The generalized entropies were estimated at the parameter values $q = -4.0, -3.0, -1.0, 0.0, 0.5, 1.5, 2.0, 3.0, 4.0$. Further, to interpolate for the intermediate values of $q$, we have used a specially designed quadratic splines which preserve convexity and monotonicity of the data, see [8] and [7] for the source code. Numerically differentiating the interpolating function, we obtained the Legendre transform of $T(q) = (1-q)H(q)$ and hence the estimate of the multifractal spectrum of local entropies, see 8.5.b. It is known that the straight line through the origin with a slope $45^o$ is tangent to the graph of the spectrum of local entropies at the point $\alpha_1$, equal to the value of the measure-theoretic en-




(a): The graph of the Manneville-Pomeau map with $s = 0.9$, $x = 0$ is an indifferent fixed point.

(b): One orbit of the Maneville-Pomeau map, intermittent behavior is clearly visible.

Figure 8.6: Manneville-Pomeau map with $s = 0.9$.

tropy. Hence by drawing such a line and determining the corresponding value $\hat{\alpha}_1$, we obtain an estimate of the measure-theoretic entropy as well.

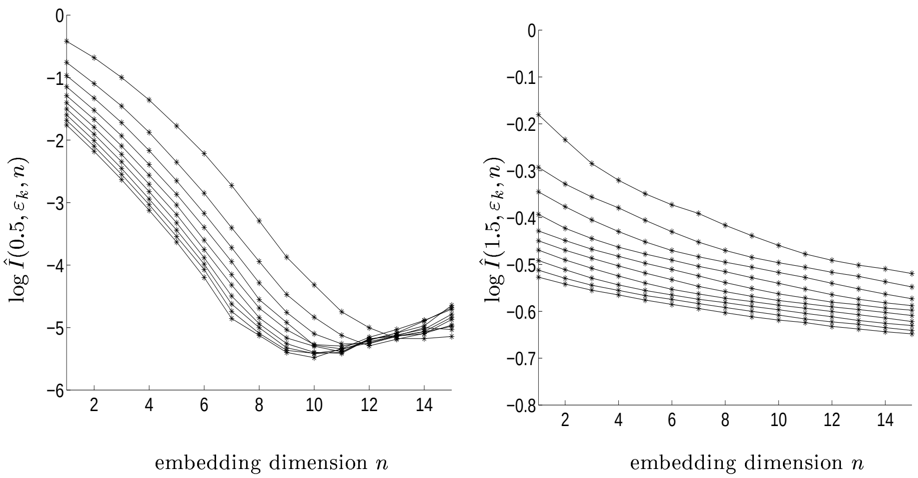
### 8.2.2 Manneville-Pomeau map

The Manneville-Pomeau maps have been proposed as the simplest models for intermittent behavior. The Manneville-Pomeau maps are piecewise continuous transformations of the unit interval given by

$$f_s(x) = x + x^{1+s} \bmod 1,$$

where $s > 0$. We are interested in the case $s \in (0,1)$. For these values of the parameter $s$ the Manneville-Pomeau map $f_s$ admits a finite absolutely continuous invariant measure. In the case of the Manneville-Pomeau maps, we do not have analytic expressions for the generalized entropies for all $q$, and hence we have nothing to compare our results with. However, at one point $q$ we can: the correlation entropy of order 0 must be equal to the topological entropy $h_{top}(f) = \log 2$. The main interest in estimating the generalized entropies for the Manneville-Pomeau maps is to see if our methods can detect the discontinuity (phase transition) occurring in the family of correlation entropies at $q = 1$. Namely, $H(q) \equiv 0$ for all $q > 1$, and $H(q) \geq c > 0$ for all $q \leq 1$.

Estimates of $\log \hat{I}(q, \varepsilon, n)/(q-1)$ for $q = 0.5$ and $q = 1.5$ are represented in figure 8.7. It is clear, that for $q = 1.5$ the decay is slower than linear. The estimate for $q = 0.5$, figure 8.7.a, needs some explanation. We see, that the




(a) Case $q = 0.5$, non-monotonicity of the estimates is explained in the text.

(b) Case $q = 1.5$, slow decay is clearly visible.

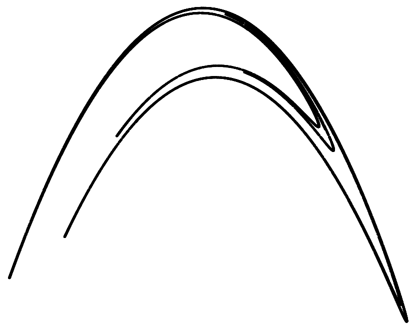
Figure 8.7: Estimates of $\log \hat{I}(q, \varepsilon_k, n)$ for $q = 0.5$ and $q = 1.5$.

estimates of $\log \hat{I}(q, \varepsilon, n)/(q-1)$ are not monotonic in $n$, while the true values of $\log I(q, \varepsilon, n)/(q-1)$ are monotonic. This is a drawback of the formula (8.2) which is used as an estimate for $I(q, \varepsilon, n)$. Namely, for $q < 1$ we have

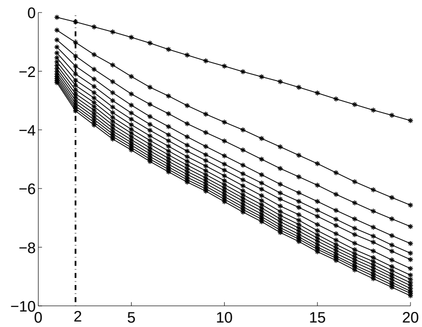
$$\hat{I}(q, \varepsilon, n) = \frac{1}{N_{ref}} \sum_{k=1}^{N_{ref}} {}^{*} \big( C(x_{i(k)}, \varepsilon, n) \big)^{q-1}, \tag{8.6}$$

where $\sum^*$ is the summation over all $k$ such that $C(x_{i(k)}, \varepsilon, n) > 0$. The explanation of the non-monotonicity is as follows: despite the fact $C(x_{i(k)}, \varepsilon, n)$ are monotonically decreasing in $n$ for all $x_{i(k)}$ and $\varepsilon > 0$, the estimate (8.6) might not be monotonic since we disregard all $k$ with $C(x_{i(k)}, \varepsilon, n) = 0$. Apparently, this is what happens in the case of the Manneville-Pomeau maps. We have also observed a similar phenomenon in the case of the Hénon map, discussed below. It is important to mention that we have not observed this problem during the computation of the generalized entropies in the case of the skew-tent map. This can be explained as follows: for the skew-tent map and for uniformly hyperbolic dynamical systems in general, the decay of $\mu(\mathcal{B}_n(x, \varepsilon))$ with respect to $n$ is uniform, and the rate of this decay is bounded from above and from below by positive numbers. For non-uniformly hyperbolic dynamical systems this is not the case, leading to the discussed problem of non-monotonicity of our estimators. In our opinion, this "unpleasant" property can be used for a practical discrimination between hyperbolic and non-hyperbolic systems.

Finally, using the results represented in figure 8.7, we obtained the following



(a): Reconstruction of the Hénon attractor from a time series $\{x_i\}$.




(b): Estimates of $\log \hat{I}(2, \varepsilon_k, n)$ for different $\varepsilon_k$. Notice changes in slopes at $n = 2$.

Figure 8.8: Hénon map with parameters $a = 1.4$, $b = 0.3$

estimates of the correlation entropies of orders $q = 0.5$ and $q = 1.5$:

$$\hat{H}(0.5) = 0.4492, \quad \hat{H}(1.5) = 0.008.$$

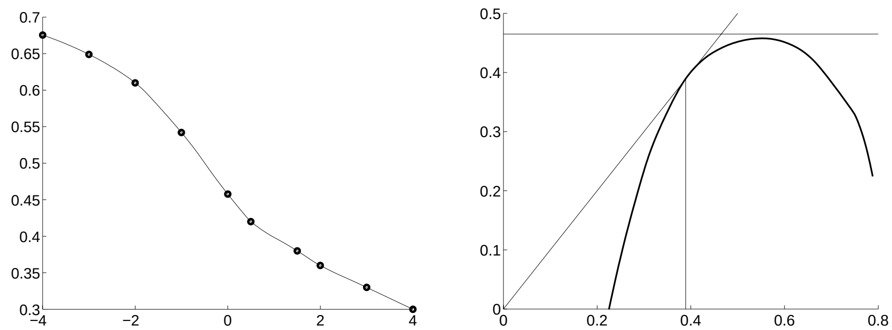
Hence we can say that our method is capable of detecting phase transitions in the family of generalized entropies.

### 8.2.3 Hénon map

The Hénon map is a transformation of the plane $\mathbb{R}^2$ into itself, given by

$$\begin{aligned} x_{i+1} &= 1 - a x_i^2 + y_i, \\ y_{i+1} &= b x_i. \end{aligned}$$

We study the Hénon family for the so-called *standard parameters* $a = 1.4$ and $b = 0.3$. The time series is obtained by recording only the first coordinate $x_i$. Figure 8.8.a shows a reconstruction of the Hénon attractor, and the estimates of $-\log \hat{I}(2, \varepsilon_k, n)$ for different $\varepsilon_k$ are represented in figure 8.8.b. Notice, how the slope is changing at $n = 2$. This is a consequence of the fact that $n = 2$ is a sufficient embedding dimension for the reconstruction of the Hénon attractor. Hence we have to use the values $-\log \hat{I}(q, \varepsilon_k, n)$ with $n \geq 2$ for subsequent estimation of the correlation entropies. Results of our estimation of the correlation entropies and subsequent estimate of the multifractal spectrum of local entropies are represented in figure 8.9. We found that our estimate of the topological entropy is quite close to the estimate obtained in [9]. We also find estimation of the generalized entropies for the Hénon family a much more difficult task than the estimation for the skew-tent map or the Manneville-Pomeau map: estimates of the slopes tend to oscillate longer and with higher amplitudes. However this should be expected form a non-hyperbolic system.




(a): Generalized entropies of the the Hénon map with $a = 1.4$, $b = 0.3$, for $q = -4.0, \dots, 4.0$. Estimated values are represented by circles, solid line was obtained using quadratic splines from [8].

(b): Estimate of the the multifractal spectrum of local entropies for the Hénon map. Tangent line gives an estimate of the measure-theoretic entropy $\hat{h} = 0.3890$; the hight of the horizontal line represents an estimate of the topological entropy $\hat{h}_{top} = 0.4645$ obtained in [9], the maximum of the graph should also agree with the topological entropy.

Figure 8.9: Estimates of the correlation entropies and the multifractal spectrum for local entropies in the case of the Hénon map.

## 8.3 Gaussian kernel approach.

One can easily see that

$$\mu(\mathcal{B}_n(x,\varepsilon)) = \int \Theta\big(\varepsilon - d_n(x,y)\big)\, d\mu(y),$$

where $\Theta(\cdot)$ is the Hevesaide function

$$\Theta(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It was suggested in the literature [5, 4, 2] that a different, somewhat smoother kernel can give better results. The basic properties, required from a such kernel function $F = F_\varepsilon$, are as follows:

- $F_\varepsilon(0) = 10$, $F'_\varepsilon(0) = 0$;
- $F_\varepsilon(t) = O(1)$ for $t = O(\varepsilon)$;
- $F_\varepsilon(t) \to 0$ sufficiently fast with $|t| \to \infty$.

A natural candidate is the Gaussian kernel

$$F_\varepsilon(t) = \exp\Big( - \frac{t^2}{2\varepsilon^2} \Big).$$

The normalizing constant is not important, since any constant will disappear in the subsequent estimates of the correlation entropies. Using the Gaussian kernel, we obtain the following estimate

$$\hat{I}^g(q,\varepsilon,n) = \frac{1}{N_{ref}} \sum_{k=1}^{N_{ref}} \left[ \frac{1}{N-n-1} \sum_{j \neq i(k)} \exp\left( - \frac{d_n^2(x_j, x_{i(k)})}{2\varepsilon^2} \right) \right]^{q-1}.$$

Now we do not have to distinguish cases $q < 1$ and $q > 1$ since the expression in square brackets is always positive. The hypothesis is that $\hat{I}^g(q,\varepsilon,n)$ and $\hat{I}(q,\varepsilon,n)$, defined above, behave in a similar way, but, due to the smoother kernel, $\hat{I}^g(q,\varepsilon,n)$ will be a better estimate, since the points $x_i$, $x_j$ with $d_n(x_i,x_j)$ slightly larger than $\varepsilon$ were disregarded in the definition $\hat{I}(q,\varepsilon,n)$, but accounted for in $\hat{I}^g(q,\varepsilon,n)$. However the numerical results (figure 8.10) show that this does not have such a big effect, and does not improve our final estimates.

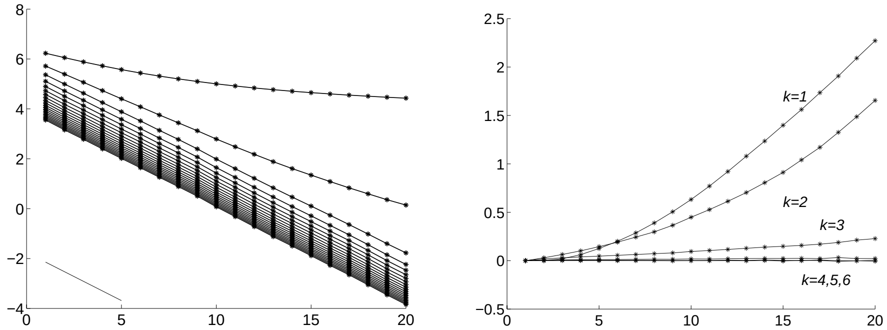
Figure 8.10.a shows the results of the computation for the skew-tent map with the same parameters, used in the computation of the results presented in figure 8.1. One can easily see that for the curves, corresponding to $\varepsilon_1$ and $\varepsilon_2$ (due to monotonicity the curve, corresponding to $\varepsilon_k$, is the $k$-th curve from the top), are slightly bended upwards in comparison with the same curves in figure 8.1. This is an effect of a Gaussian kernel which has a heavier tail than that one of the Hevesaide kernel. For smaller values $\varepsilon_k$, $k > 2$, the bending is not visible with the naked eye, and indeed, as the figure 8.10.b shows, the effect of such bending is very small for $k > 3$. Hence for sufficiently small $\varepsilon > 0$, the graphs of $\log \hat{I}(2,\varepsilon,n)$ and $\log \hat{I}^g(2,\varepsilon,n)$ as functions of $n$, are almost parallel. Thus, using the least squares for estimation the slope will produce the same result. However, numerical estimates based on the Gaussian kernel are more expensive in terms of computing time.

Another suggestion was to substitute the maximum norm

$$d_n(x,y) = \max_{i=0,\dots,n-1} d\big(f^i(x), f^i(y)\big).$$

in the definition of the correlation entropies by the Euclidian norm

$$d_n^E(x,y) = \Big( \sum_{i=0}^{n-1} d^2\big(f^i(x), f^i(y)\big) \Big)^{\frac{1}{2}}.$$




(a): Graphs of $\log \hat{I}^g(2, \varepsilon_k, n)$ for $\varepsilon_k = (2k)^{-2}$, $k = 1, \dots, 20$, as functions of $n$. The slope of a short straight line in the lower left corner is equal to the theoretical value of the correlation entropy of order 2.

(b): Graphs of $\log \hat{I}^g(2, \varepsilon_k, n) - \log \hat{I}(2, \varepsilon_k, n) - \log \hat{I}^g(2, \varepsilon_k, 1) + \log \hat{I}(2, \varepsilon_k, n)$ as a function of $n$ for $k = 1, \dots, 6$.

Figure 8.10: Estimation of the correlation entropy of order $q = 2.0$ with the Gaussian kernel for the skew-tent map, parameters are the same as in figure 8.1.

However, we found that this leads to a bias in the estimates, and indeed a correction for $q = 2.0$ has been suggested in the literature [4, 2]. Namely, using the Euclidian norm instead of the maximum norm, one has to base estimators on the assumption that

$$I^E(2.0, \varepsilon, n) \approx Ce^{-H(2)n} \left( \frac{1}{\sqrt{n}} \right)^{D(2)},$$

where $D(2)$ is the generalized dimension of $\mu$ of order 2.

## 8.4 Concluding remarks

In the present chapter we have suggested several estimators of the generalized entropies and demonstrated their properties using a few model examples. The methods, we have developed, produce estimators with a quality not worse than those reported in the physics literature, even for a time series of moderate lengths: $N$ is of the order of a few thousands. Also, our methods are more economical with respect to memory usage, when compared to the estimation schemes for the Rényi entropies, see [6].

Secondly, graphical representation of the data like in figures 8.1, allows a quick and simple choice of the relevant parameters such as $\varepsilon$ and the embedding dimensions $n$ which are necessary for the estimation.

In the present chapter we did not address statistical problems arising in the estimation of the generalized entropies. These problems seem to be quite interesting and difficult, and, in our opinion, deserve further investigation. We would like to mention that there are a lot of similarities between these problems and the so-called Hill estimator. The latter attracted and still attracts a lot of attention from the researches in probability theory and mathematical statistics, see [3] and references therein.

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# Summary

Researchers working in various areas such as biology, chemistry, physics and economics are often facing a similar problem of extracting information and making conclusions about the dynamics of complex systems, based on the knowledge of one or several realizations of such a system.

One of the typical questions arising in relation to this problem is, whether a seemingly chaotic time series is indeed chaotic and how chaotic is it? Since the works of C. Shannon on information theory, a traditional way of evaluating complexity is assigning a specific non-negative number to the source of data, called the *entropy*. In dynamical systems the notion of entropy was further developed by A.N. Kolmogorov and Ya.G. Sinai in the sixties. The so-called measure–theoretic entropy plays an important role in the description of dynamical systems when they are viewed from a probabilistic point of view. It is also clear that, despite a widely accepted understanding of entropy as one of the most important characteristics of dynamical systems, for many purposes assigning just one number to a complex system cannot be adequate.

A. Rényi, in the early seventies, proposed an infinite number of entropy characteristics, generalizing the Shannon entropy. These quantities, now known as the Rényi entropies, became widely accepted in information theory, probability theory and mathematical statistics. The main idea of Rényi was that the standard Shannon entropy is obtained as a certain average, and by using different averaging procedures, we, in principle, can get new types of entropies. The question is of course what are the meaningful averaging procedures which one should use. Rényi has solved this problem in an elegant way. He devised an axiomatic approach to entropy: by choosing a small number of useful properties of the Shannon entopy and taking these as axioms. A remarkable result of Rényi's is that the (generalized) entropies, which satisfy these axioms, form a smooth one-parameter family.

The Rényi entropies were brought into dynamical systems by H.G.E. Hentschel and I. Procaccia in 1983. They argued that the Rényi entropies will provide new information about dynamical systems, and in particular, they will be useful for the purposes of the multifractal analysis. In approximately the same time, using a similar motivation, F. Takens proposed a different family of generalized entropies. His approach was motivated by the so-called *Reconstruction Theorem*, which states that under some mild assumptions certain aspects of a dynamical system, like its dimension and entropy, can be recovered from one sufficiently long time series.

The family of generalized entropies proposed by F. Takens is the main subject of this thesis.

We have to mention that the ideas of Rényi to use different ways of averaging were not applied to entropies only, but to dimensions as well. Also, at the moment when the research, presented in this thesis, began, the theory of generalized dimensions was developed much further than the corresponding theory of entropies. One of the main goals of the research was to bring the theory of generalized entropies to a similar level. Throughout this thesis we compare analogous results for dimensions and entropies. From these comparisons we see that the corresponding results for entropies are valid for a larger class of dynamical systems than those for dimensions. For example, the multifractal formalism for local entropies is valid for hyperbolic diffeomorphisms on manifolds of arbitrary dimension, while the corresponding multifractal formalism for local dimensions is established only for hyperbolic systems in dimensions 1 and 2. Moreover, most probably, the multifractal formalism for local dimensions is not valid for all hyperbolic systems in dimensions 3 and higher. This allows us to claim that the theory for entropies is more natural, in a certain sense, than the corresponding theory for dimensions.

This thesis is organized as follows. In the introduction we give an overview of the basic concepts of dynamical systems such as the notions of attractors, hyperbolicity, physical (SRB) measures. We give definitions of local (pointwise) dimensions and entropies, and also we define the families of generalized dimensions and entropies, which are global quantities. The fundamental relation between local and global dimensions or entropies, is known as the *Multifractal Formalism.* We cite the results on the multifractal formalism for dimensions and compare them with the analogous results (established in the thesis) for entropies. One of the main results of this thesis (chapter 5) implies the validity of the multifractal formalism for local entropies for expansive homeomorphisms satisfying the specification property and their invariant Gibbs measures. This class of transformations is not as widely used as, for example, the more restrictive class of hyperbolic dynamical systems. There are certain similarities and essential differences between these classes of dynamical systems. The last part of the introduction is devoted to a comparison between them.

In the second chapter we establish the basic properties such as monotonicity and continuity of the family of the generalized entropies proposed by F. Takens. We also obtain explicit expressions for the generalized entropies of symbolic dynamical systems in the case of Bernoulli and Gibbs measures. Possible singularities in the family of generalized entropies – the so-called phase-transitions, are demonstrated by two examples of non-uniformly hyperbolic interval maps.

Chapters 3 and 4 are devoted to the study of the Rényi entropies of measure-preserving dynamical systems. The main result of chapter 3 states that for ergodic dynamical systems with positive measure-theoretic entropy the Rényi entropy of order $q$ is either infinite, or is equal to the measure-theoretic entropy, depending whether $q$ is smaller or larger than 1, respectively. Hence the Rényi entropies, as measure-theoretic invariants of a dynamical system, do not provide any new information for ergodic systems with positive entropy. In the fourth chapter the assumptions of ergodicity and positivity of the measure-theoretic entropy are dropped. Then the result is slightly different: for $q < 1$, the Rényi entropies are still infinite, but for $q > 1$, the Rényi entropies are equal to the essential infimum of the measure-theoretic entropies, taken over all ergodic measures constituting the decomposition into ergodic components. The latter can be strictly smaller than the measure-theoretic entropy itself. It is surprising that the entropy-like invariants of a dynamical system can detect ergodicity. So in the non-ergodic case the situation is slightly different. However on the whole, the Rényi entropies give practically no new information. We have to mention that Hentschel and Procaccia used a similar definition of the Rényi entropies, but their implementation is equivalent to the entropies proposed by F. Takens.

In general, one should expect the multifractal formalism for local entropies to be valid without strong assumptions on the underlying dynamics and the invariant measure. In the fifth chapter we show that under very mild assumptions on the invariant measure, the Legendre transform of the generalized entropies gives an upper estimate of the multifractal spectrum of local entropies.

In the sixth chapter we discuss the most general situation when the validity of the multifractal formalism for local entropies has been established. Namely we prove this for the expansive homeomorphisms with the specification property and their invariant Gibbs measures.

The seventh chapter is dedicated to the study of the absolutely continuous invariant measures for interval maps with indifferent fixed points, the so-called Manneville-Pomeau type maps. Our interest in these dynamical systems was motivated by the fact that the generalized entropies have a singularity, namely, they are discontinuous at $q = 1$. It turns out that the absolutely continuous invariant measures for the Manneville-Pomeau maps are not Gibbs, but weakly Gibbs measures. The notion of weakly Gibbs states is currently attracting attention in Statistical Physics with regards to Dobrushin's reconstruction program.

In the last chapter we present methods for numerical estimation of the generalized entropies based on one sufficiently long orbit. We apply these methods to the time series produced by the skew-tent map, the Manneville-Pomeau map and the Hénon map. We show that for the skew-tent map, for which we know the true results from analytic considerations, one can consistently estimate the generalized entropies. For the Manneville-Pomeau map our results are well explained by the phase transition, which we already discussed. For the Hénon family our results are consistent with those reported in the literature.

# Samenvatting

Onderzoekers in verschillende gebieden, zoals biologie, scheikunde, natuurkunde of economie, lopen vaak tegen een soortgelijk probleem aan: het extraheren van informatie en trekken van conclusies uit de dynamica van complexe systemen, aan de hand van één of meerdere realisaties van zo'n systeem. Een van de typische vragen die met betrekking tot dit probleem opkomen, is of schijnbaar chaotische tijdreeksen inderdaad chaotisch zijn, en zo ja, hoe chaotisch ze zijn. Een traditionele methode, gebaseerd op het werk van C. Shannon in de informatietheorie, is het toekennen van een bepaald getal, de *entropie*, aan de gegevensbron. Het begrip entropie is in de zestiger jaren in de context van dynamische systemen uitgewerkt door A.N. Kolmogorov en Ya.G. Sinai. De zogenaamde maat-theoretische entropie speelt sindsdien een belangrijke rol in de beschrijving van dynamische systemen, beschouwd vanuit het standpunt van de waarschijnlijkheidsrekening. Het is ook duidelijk dat, ondanks een brede concensus dat entropie één van de belangrijkste karakteristieken van een dynamisch systeem is, voor veel toepassingen het beschrijven van een complex systeem door slechts één getal niet toereikend kan zijn.

In de zeventiger jaren introduceerde A. Rényi een oneindig aantal entropie-karakteristieken, die de Shannon-entropie generaliseerden. Deze karakteristieken, nu bekend als Rényi-entropieën, werden algemeen geaccepteerd in de informatietheorie, waarschijnlijkheidsrekening en mathematische statistiek. Het belangrijkste idee van Rényi was dat de gewone Shannon-entropie verkregen wordt als een zeker gemiddelde, en dat door verschillende methodes van middelen te gebruiken we, in principe, nieuwe types entropie kunnen krijgen. De vraag is nu natuurlijk wat de zinvolle methodes van middelen zijn. Rényi heeft dit probleem op een elegante manier opgelost. Hij ontwierp een axiomatische benadering van entropie. Een klein aantal bruikbare eigenschappen van de Shannon-entropie vormde de basis van Rényi's axioma's. Een opmerkelijk resultaat van Rényi is dat er een unieke één-parameter familie van gegeneraliseerde entropieën bestaat, die aan deze axioma's voldoen.

De Rényi-entropieën zijn ingevoerd in dynamische systemen door H.G.E. Hentschel en I. Procaccia in 1983. Zij betoogden dat de Rényi-entropieën nieuwe informatie over dynamische systemen zouden geven, en in het bijzonder, dat zij bruikbaar zouden zijn in multifractale analyse. Ongeveer rond dezelfde tijd, en met een soortgelijke motivatie, stelde F. Takens een andere familie van gegeneraliseerde entropieën voor. Zijn aanpak was gemotiveerd door de zogenaamde Reconstructiestelling, die zegt dat, onder milde voorwaarden, bepaalde aspecten van een dynamisch systeem, zoals dimensie en entropie, teruggevonden kunnen worden uit één voldoend lange tijdreeks. Deze familie van gegeneraliseerde entropieën voorgesteld door F. Takens is het hoofdonderwerp van dit proefschrift.

We moeten hier opmerken dat de ideeën van Rényi om verschillende manieren van middelen te gebruiken niet alleen op entropieën werden toegepast, maar ook op dimensies. Op het moment dat het onderzoek, gepresenteerd in dit proefschrift, begon, was de theorie van gegeneraliseerde dimensies veel verder ontwikkeld dan de corresponderende theorie van entropieën. Eén van de hoofddoelstellingen van het onderzoek was om de theorie van gegeneraliseerde entropieën op een vergelijkbaar niveau te krijgen. Het hele proefschrift door vergelijken we analoge resultaten voor dimensies en entropieën. Uit deze vergelijkingen blijkt dat de corresponderende resultaten geldig zijn voor een grotere klasse van dynamische systemen dan die voor dimensies. Bijvoorbeeld, de multifractale formalismen voor lokale entropieën zijn geldig voor hyperbolische diffeomorfismen op variëteiten van willekeurige dimensie, terwijl het corresponderende multifractale formalisme voor lokale dimensies slechts is bewezen voor hyperbolische systemen in dimensie 1 en 2. Hoogstwaarschijnlijk is het multifractale formalisme voor lokale dimensies niet geldig voor alle hyperbolische systemen in dimensie 3 en hoger. Dit rechtvaardigt de bewering dat de theorie van entropieën in zekere zin *natuurlijker* is dan de corresponderende theorie voor dimensies.

De opbouw van dit proefschrift is als volgt. In de inleiding geven we een overzicht van de basisconcepten van dynamische systemen, zoals attractoren, hyperboliciteit en natuurlijke (SRB) maten. We geven definities van lokale (puntsgewijze) dimensies en entropieën, en families van gegeneraliseerde dimensies en entropieën, die globale karakteristieken zijn. Het fundamentele verband tussen de lokale en globale dimensies of entropieën staat bekend als het *multifractale formalisme*. We halen resultaten aan over het multifractale formalisme voor dimensies, en vergelijken deze met de analoge resultaten (bewezen in dit proefschrift) voor entropieën. Een van de belangrijkste resultaten (hoofdstuk 5) impliceert de geldigheid van het multifractale formalisme voor lokale entropieën voor expansieve homeomorfismen die de specificatie-eigenschap hebben. Deze klasse van transformaties wordt minder vaak gebruikt dan, bijvoorbeeld, de klasse van hyperbolische dynamische systemen. Er zijn zekere overeenkomsten, maar ook essentiële verschillen, tussen deze klassen van dynamische systemen. Het laatste deel van de inleiding is gewijd aan een vergelijking tussen deze klassen.

In het tweede hoofdstuk bewijzen we de basiseigenschappen van de familie van gegeneraliseerde entropieën van F. Takens, zoals monotoniciteit en continuïteit. We geven ook expliciete uitdrukkingen voor de gegeneraliseerde entropieën van symbolische dynamische systemen in het geval van Bernoulli– en Gibbsmaten. Mogelijke singulariteiten in de familie van gegeneraliseerde entropieën—de zogenaamde faseovergangen—worden gedemonstreerd met twee voorbeelden van niet-uniforme hyperbolische intervalafbeeldingen.

Hoofdstukken 3 en 4 zijn gewijd aan Rényi-entropieën van maatbehoudende dynamische systemen. In hoofdstuk 3 wordt aangetoond dat voor ergodische dynamische systemen met positieve maat-theoretische entropie, de Rényi-entropie van orde $q$ hetzij oneindig, hetzij gelijk is aan de maat-theoretische entropie, wanneer $q$ kleiner, respectievelijk groter is dan 1.

Hieruit volgt dat de Rényi-entropieën, in hun rol als maat-theoretische invarianten van een dynamisch systeem, geen nieuwe informatie geven voor ergodische systemen met positieve entropie. In het vierde hoofdstuk laten we aannames over ergodiciteit en het positief zijn van de maat-theoretische entropie vallen. Dit leidt tot een iets ander resultaat: voor $q < 1$ zijn de Rényi-entropieën nog steeds oneindig, maar voor $q > 1$ zijn ze gelijk aan het essentiële infimum van de maat-theoretische entropieën, genomen over alle ergodische maten die de decompositie in ergodische componenten vormen. Dat laatste kan strikt kleiner zijn dan de maat-theoretische entropie zelf. Het is verrassend dat de entropie-achtige invariant van een dynamisch systeem ergodiciteit kan detecteren. In het niet-ergodische geval is de situatie dus enigszins anders. Echter, we kunnen zeggen dat de Rényi-entropieën nauwelijks nieuwe informatie geven. We moeten hier opmerken dat Hentschel en Procaccia een gelijksoortige definitie van Rényi-entropieën gebruikten, maar hun implementatie is equivalent met de entropieën zoals voorgesteld door F. Takens.

In het algemeen zou men verwachten dat het multifractale formalisme voor lokale entropieën geldig is zonder sterke voorwaarden op de onderliggende dynamica en de invariante maat. In het vijfde hoofdstuk laten we zien dat onder zeer milde voorwaarden op de invariante maat, de Legendre-getransformeerde van de gegeneraliseerde entropie een bovengrens geeft van het multifractale spectrum van lokale entropieën.

In het zesde hoofdstuk beschouwen we de meest algemene situatie waarin de geldigheid van het multifractale formalisme voor lokale entropieën is bewezen. We bewijzen het namelijk voor expansieve homeomorfismen met de specificatie-eigenschap en hun invariante Gibbsmaten.

Het zevende hoofdstuk is gewijd aan het onderzoek van absoluut continue invariante maten voor intervalafbeeldingen met indifferente vaste punten, de zogenaamde Manneville-Pomeau afbeeldingen. Onze interesse in deze dynamische systemen kwam voort uit het feit dat de gegeneraliseerde entropieën een singulariteit hebben. Ze zijn namelijk discontinu bij $q = 1$. Het blijkt dat de absoluut continue invariante maten voor de Manneville-Pomeau afbeeldingen niet Gibbs zijn, maar zwak-Gibbs. Het begrip zwak-Gibbs trekt op dit moment veel aandacht in de statistische fysica, in het kader van Dobrushin's reconstructieprogramma.

In het laatste hoofdstuk presenteren we methoden voor de numerieke benadering van gegeneraliseerde entropieën, gebaseerd op een voldoend lange tijdreeks. We passen deze methoden toe op tijdreeken geproduceerd door de scheve tent afbeelding, de Manneville-Pomeau afbeelding en de Hénon-afbeelding. We laten zien dat voor de scheve tent afbeelding, waarvoor analytische resultaten bekend zijn, de gegeneraliseerde entropieën op een consistente manier geschat kunnen worden. Voor de Manneville-Pomeau afbeelding worden onze resultaten goed verklaard door de faseovergang, die we hiervoor al bespraken. Voor de Hénon-familie zijn onze resultaten in overeenstemming met resultaten in de literatuur.